Interest Rates and Time Value of Money

MATH 372 *Financial Mathematics I*

J. Robert Buchanan

Department of Mathematics

Fall 2021

Interest

Definition

Interest is a form of rent paid by a borrower of money to the lender for the use of the money over a period of time.

Remark: interest rates (rent per unit time) vary between lenders. One standard is known as the London InterBank Offered Rate (LIBOR).

Terminology

PV: present value, typically the amount borrowed or lent.

FV: future value, present value plus interest.

i: annual interest rate (expressed as a decimal).

j: interest rate for a period other than a year.

Definition

Interest is called **simple** if it is earned only on the present value. Interest is called **compound** if it is earned on the total amount of the account at the beginning of a time period.

Simple *vs.* Compound Interest

For **simple interest** calculations,

$$\mathrm{FV} = \mathrm{PV}(1 + i\,t)$$

where t is measured in years.

For **compound interest** calculations,

$$\mathrm{FV} = \mathrm{PV}(1 + i)^t$$

if the interest rate remains constant for t years, and

$$\mathrm{FV} = \mathrm{PV}(1 + i_1)(1 + i_2)\cdots(1 + i_t)$$

if the interest rate is i_1 in year 1, i_2 in year 2, ..., i_t in year t.

Relationship Between PV and FV

Present value and future value of an investment are always related through the following two equivalent equations.

$$\text{FV} = \text{PV}(1+i)^n$$
$$\text{PV} = \frac{\text{FV}}{(1+i)^n}$$

The BA II Plus calculator has special keys for working with PV, FV, i, and n. You must become familiar with them. See http://www.youtube.com/v/sF-91SoT7lU.

More

Example

Let $n = 10$ and $i = 6\%$ (interest compounded annually for 10 years at 6% per year).

1. If $\text{PV} = 1000$, find FV.
2. If $\text{FV} = 1000$, find PV.

More

Example

Using an interest rate of 5% compounded annually, find

1. the present value of \$20,000 payable in 15 years, and
2. the future value of \$5,000 in 6 years.

Effective Rates of Interest

Definition

The **effective annual rate of interest** earned by an investment during a one-year period is the percentage change in the value of the investment from the beginning to the end of the year, without regard to the investment behavior at intermediate points in the year.

Definition

Two rates of interest are **equivalent** if they result in the same accumulated values at each point in time.

Accumulation Factor

Definition

Let $a(t)$ be the accumulated value at time t of an investment of \$1 made at time 0 and defined as the **accumulation factor** from time 0 to time t. Let $A(t)$ denote the accumulated amount of an investment at time t, so that if the initial investment is $A(0)$, then the accumulated value at time t is

$$A(t) = A(0)a(t).$$

- At effective annual rate of interest i per period, the accumulation factor from time 0 to time t is

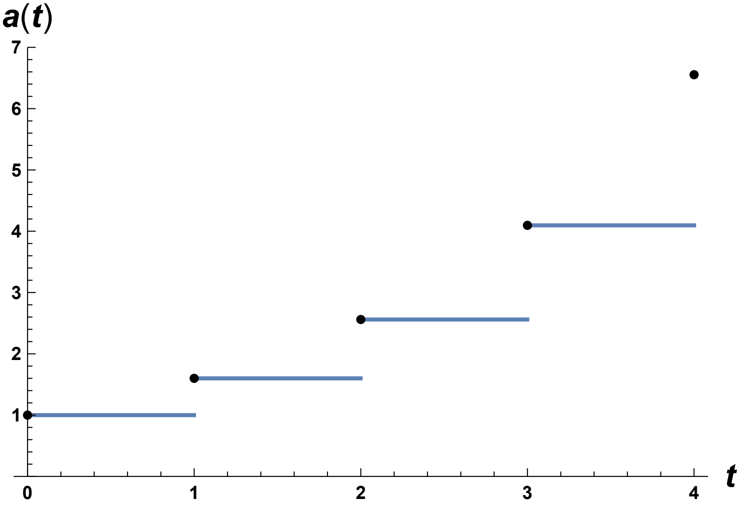
$$a(t) = (1+i)^t.$$

- For annual simple interest rate i, the accumulation factor from time 0 to time t is

$$a(t) = 1 + i\,t.$$

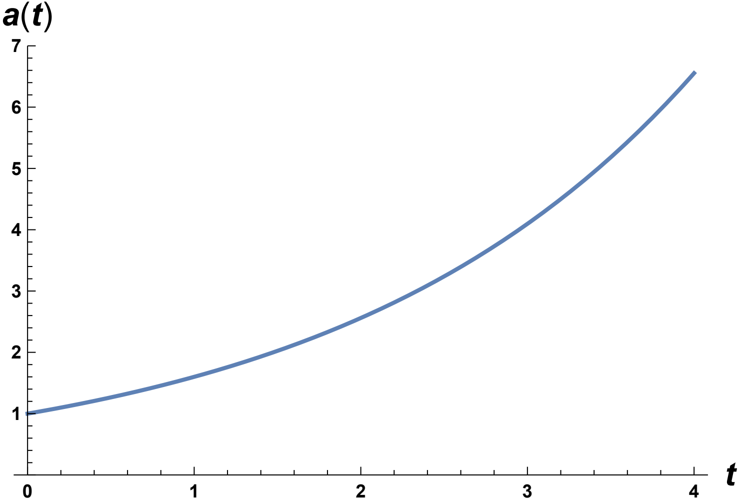
When is Interest Paid?

If interest is earned only at the end of each year (month, week, *etc.*) then $a(t)$ is a step function.



Interest Paid Continuously

Unless specifically told that interest is paid only at the end of each year (month, week, *etc.*) assume $a(t)$ is defined for all t.



Effective Rate of Interest

Notation: $i_{[t,t+1]}$ denotes the effective annual interest rate for the one-year period from time t to time $t+1$.

$$i_{[t,t+1]} = \frac{A(t+1) - A(t)}{A(t)} = \frac{a(t+1) - a(t)}{a(t)} = \frac{\text{interest earned}}{\text{value at start}}$$

Remark: the briefer notation i_{t+1} is sometimes used instead and should be thought of as the interest rate for year $t+1$.

Examples

Let the interest rate be 6% and the time interval be $[2, 3]$.

1. Find $i_{[2,3]}$ assuming compound interest.
2. Find $i_{[2,3]}$ assuming simple interest.

Nominal Rate of Interest

Definition

A **nominal annual rate of interest compounded (convertible)** m **times per year** refers to an interest compounding of $1/m$ years. The interest rate for a time period of $1/m$ years is

$$\frac{\text{nominal annual rate}}{m}.$$

Actuarial Notation

i: effective annual rate

$i^{(m)}$: nominal annual rate with interest compounded m times per year

$\frac{i^{(m)}}{m}$: periodic rate

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

$$i^{(m)} = m\left((1 + i)^{1/m} - 1\right)$$

Example

Tom is trying to decide between two banks in which to open a saving account. Bank A offers an annual rate of 15.25% with interest compounded semiannually, and Bank B offers an annual rate of 15% with interest compounded monthly. Which bank has the higher effective interest rate?

The video at
https://www.youtube.com/watch?v=HqWVvsrmPWE
illustrates two methods for solving this type of problem on the calculator.

Rate of Discount

There are many ways to structure investments. Suppose an investor would like to earn 5% for one year. They could

- Invest \$1000 at the beginning of the year and receive a payment of \$1050 at the end of the year.
- Target a payment of \$1000 at the end of the year and “discount” that amount to determine how much to invest.

$$\text{PV} = \frac{1000}{1.05} = \$952.38$$

The difference $1000 - 952.38 = 47.62$ is called the **discount** and

$$\frac{47.62}{1000} = 0.04762 = d$$

is the **rate of discount**.

Relationship Between *i* and *d*

Definition

The **annual effective rate of discount** is the amount of interest earned during the year divided by the ending balance.

$$
\begin{aligned}
d &= \frac{A(t+1) - A(t)}{A(t+1)} \\
&= \frac{a(t+1) - a(t)}{a(t+1)} \\
&= \frac{(1+i)^{t+1} - (1+i)^{t}}{(1+i)^{t+1}} \\
d &= 1 - \frac{1}{1+i} = \frac{i}{1+i}
\end{aligned}
$$

Example

- For $i = 0.075$ find d.

- For $d = 0.09$ find i.

Effective Annual Rate of Discount

In some loan situations, the interest is paid at the beginning of the loan period (rather than at the end).

Example

Suppose Smith takes out a loan of \$1000 for one year at an interest rate of 10% *payable in advance*.

- At $t = 0$ Smith receives loan amount of \$1000 and must immediately pay \$100 in interest. The loan is effectively \$900.
- At time $t = 1$ Smith repays principal amount of \$1000.

What is the equivalent interest rate for interest payable at the end of the loan period?

Present Value Factor

Definition

If the rate of interest for a period is i, the present value of \$1 due one period from now is

$$\frac{\$1}{1+i} = v$$

in actuarial notation and is called a **present value factor.**

$$\begin{aligned} A(t) &= A(0)(1+i)^t \\ A(0) &= A(t)v^t \end{aligned}$$

Relationships Between d, i, and v

$$
\begin{aligned}
v &= \frac{1}{1+i} \\
d &= \frac{i}{1+i} = i\,v \\
d &= 1 - \frac{1}{1+i} = 1 - v \\
i - d &= i - i\,v = i(1-v) = i\,d
\end{aligned}
$$

Example

Given that $d = 0.065$ find i and v.

Nominal Annual Discount Rate

Definition

A **nominal annual rate of discount compounded** m **times per year** refers to a discount compounding period of $1/m$ years. The discount rate for a period of $1/m$ years is

$$\frac{\text{nominal annual discount rate}}{m}.$$

d: effective annual discount rate

$d^{(m)}$: nominal annual discount rate with discount compounded (convertible) m times per year.

$$1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m = v = \frac{1}{1+i}$$

Example

Find the effective annual discount rate for a nominal annual interest rate of 8% convertible quarterly.

Additional Conversions

Occasionally we must convert a nominal rate of interest convertible m times per year to an equivalent nominal discount rate convertible p times per year.

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(p)}}{p}\right)^{-p}$$

The equation above is equivalent to the following equation.

$$\left(1 + \frac{i^{(m)}}{m}\right)^m \left(1 - \frac{d^{(p)}}{p}\right)^p = 1$$

Example

Find the rate of discount convertible quarterly that is equivalent to a nominal rate of interest of 10% compounded weekly.

Force of Interest

Definition

For an investment that grows according to the accumulated amount function $A(t)$, the **force of interest** at time t is

$$\delta(t) = \frac{A'(t)}{A(t)}.$$

Derivation

$$
\begin{aligned}
\delta(t) &= \frac{A'(t)}{A(t)} = \frac{d}{dt}\left[\ln A(t)\right] \\
\delta(t)\,dt &= d\left[\ln A(t)\right] \\
\int_{n_1}^{n_2} \delta(t)\,dt &= \int_{n_1}^{n_2} d\left[\ln A(t)\right] = \ln A(n_2) - \ln A(n_1) = \ln \frac{A(n_2)}{A(n_1)} \\
e^{\int_{n_1}^{n_2} \delta(t)\,dt} &= \frac{A(n_2)}{A(n_1)} \\
A(n_2) &= A(n_1) e^{\int_{n_1}^{n_2} \delta(t)\,dt}
\end{aligned}
$$

Summary

$$
\begin{aligned}
A(n_2) &= A(n_1)e^{\int_{n_1}^{n_2} \delta(t)\, dt} \\
A(n_1) &= A(n_2)e^{-\int_{n_1}^{n_2} \delta(t)\, dt}
\end{aligned}
$$

$e^{\int_{n_1}^{n_2} \delta(t)\, dt}$ is the general future value factor.

$e^{-\int_{n_1}^{n_2} \delta(t)\, dt}$ is the general present value factor.

We can express the accumulation function for varying interest rates as

$$a(t) = e^{\int_0^t \delta(s)\, ds}.$$

Example

Given that $a(t) = (2t + 1)^8$, find an expression for $\delta(t)$.

Example

Given that $\delta(t) = \frac{4}{t+1}$, find an expression for $a(t)$.

Constant Force of Interest

If $\delta(t) = \delta$ (a constant) then

$$A(t) = A(0)e^{\int_0^t \delta\, ds} = A(0)e^{\delta\, t}$$

and likewise

$$A(0) = A(t)e^{-\delta\, t}.$$

This is the case of **continuously compounded interest.**

Relationship Between δ and i

Recall that $a(t) = e^{\delta t}$, so

$$\begin{aligned} a(1) &= e^{\delta} \\ 1 + i &= e^{\delta} \\ v &= e^{-\delta} \\ \delta &= \ln(1 + i) \\ i &= e^{\delta} - 1 \end{aligned}$$

United States Treasury Bills

- US Treasury bills (T-bills) are loans an investor can make to the US government for 4 weeks, 13 weeks, 26 weeks, or 52 weeks.
- On the T-bill's maturity date the investor will receive the face value of the bill.

Example

Suppose an investor pays \$950 for a T-bill maturing in 52 weeks with a face value of \$1000. The effective interest rate for this investment is

$$1000 = 950(1 + i) \implies i = 5.2632\%.$$

Quoted Rates for US T-bills

Quoted rates for US T-bills are expressed as **bank discount yield** and calculated as

$$\text{Quoted Rate} = \frac{360}{\text{Days to Maturity}} \cdot \frac{\text{Amount of Interest}}{\text{Maturity Value}}.$$

Comment: when calculating the annual effective yield, remember that a week has 7 days and a year has 365 days.

Example

A US T-bill will mature in 13 weeks for its face value of $1000. The price of the bill is $975.

1. What is the quoted rate of the T-bill?
2. What is the annual effective yield of the T-bill?

Example

A US T-bill will mature in 182 days for its face value of $1000. Its rate is quoted as 5%.

1. What is the current price of the T-bill?
2. What is the annual effective yield of the T-bill?

Government of Canada Treasury Bills

Quoted rates for Government of Canada T-bills are expressed as interest rates according to the formula,

$$\text{Quoted Rate} = \frac{365}{\text{Days to Maturity}} \cdot \frac{\text{Amount of Interest}}{\text{Current Value}}.$$

Example

A Government of Canada T-bill will mature in 52 weeks for its face value of \$1000. The price of the bill is \$950.

1. What is the quoted rate of the T-bill?
2. What is the annual effective yield of the T-bill?

Example

A Government of Canada T-bill will mature in 182 days for its face value of $1000. Its rate is quoted as 5%.

1. What is the current price of the T-bill?
2. What is the annual effective yield of the T-bill?

Comment: when calculating the annual effective yield, remember that a week has 7 days and a year has 365 days.

Relating d, δ, and i

Theorem

Suppose $m > 1$ and $i > 0$, then

$$d < d^{(m)} < \delta < i^{(m)} < i.$$

Financial Transactions

A financial transaction includes amounts **disbursed** and amounts **received.** Calculations must include the accumulated for present values of these **dated cash flows.** An equation balancing the time values of the dated cash flows is called an **equation of value** for the transaction.

Equation of Value

To formulate the equation of value, choose a reference point in time and equate the following quantities:

1. the accumulated value of all payments already disbursed plus the present value of all payments yet to be disbursed.
2. the accumulated value of all payments already received plus the present value of all payments yet to be received.

Example

Every Friday in February (7th, 14th, 21st, 28th) Walt places a bet for $1000 on credit with his off-track bookmaking service. The betting service charges an effective weekly interest rate of 8% on all credit extended (compounded weekly). Walt loses every bet and agrees to repay his debt in four installments to be paid on March 7, 14, 21, and 28. Walt pays $1100 on each of March 7, 14, and 21. How much must Walt pay on March 28 to completely repay his debt?

Homework

- Read Chapter 1
- Exercises: sample exam problems on handout