Yield Rate of an Investment

MATH 372 *Financial Mathematics I*

J. Robert Buchanan

Department of Mathematics

Fall 2021

Objectives

In this lesson we will learn to calculate

- the internal rate of return of a sequence of cash flows,
- the net present value of investments for ranking investment alternatives,
- the dollar-weighted return for investments, and
- the time-weighted return for investments.

Internal Rate of Return

The typical financial transaction involves a number of payments issued at different points in time and a number of payments received at different points in time. The **internal rate of return** (**IRR**) is the interest rate at which the compounded value of all cashflows out equals the compounded value of all the cashflows in.

Note: we will follow the convention that cashflows out are negative while cashflows in are positive.

Internal Rate of Return

Definition

Suppose a transaction has net cashflows of $C_0, C_1, \ldots, C_n$ at time $t_0, t_1, \ldots, t_n$, then the internal rate of return for the transaction is any rate of interest satisfying the equation

$$\sum_{k=0}^{n} C_k v^{t_k} = 0.$$

Remarks

- If the cashflows occur at $t = 0, 1, 2, \ldots, n$ then finding the IRR is equivalent to solving a polynomial of the form

$$C_0 + C_1 v + C_2 v^2 + \cdots + C_n v^n = 0.$$

- There are no general purpose formulas for solving polynomials of degree $n \geq 5$, therefore we will often use the calculator to numerically approximate the solution.
- We will assume $v \geq 0$ or equivalently $i \geq -1$.

Worst Case Scenario

- Suppose $C_0 < 0$ and $C_1 = C_2 = \cdots = C_n = 0$, then $i = -1$ (or -100%).
- We can ignore any solution for which $i < -1$ which implies $v \geq 0$.

Example

An investor is asked to invest \$2,500 and is promised a payment of \$1,300 in one year and \$1,350 at the end of two years. Find the IRR.

Solution

$$
\begin{aligned}
0 &= -2500 + 1300v + 1350v^2 \\
v &= \frac{-1300 \pm \sqrt{(1300)^2 - 4(1350)(-2500)}}{2(1350)} \\
v &= 0.9620 = \frac{1}{1+i} \\
i &= 0.0395
\end{aligned}
$$

Example

An investor spends \$1,000 to set up a two-year mining operation. In the first year she has revenues of \$800 and expenses of \$200. In the second year she has revenues of \$800 and expenses of \$250. Find the IRR of the investment.

Solution

$$
\begin{aligned}
-1000 + (800 - 200)v + (800 - 250)v^2 &= 0 \\
-1000 + 600v + 550v^2 &= 0 \\
v &= 0.909091 \\
i &= 0.10 \iff i = 10\%
\end{aligned}
$$

Example

An investor is asked to invest \$20,000 that will be repaid by 5 annual end-of-year payments of \$5,000. What is the yield on this investment? Use the TVM keys and also confirm the result using the CF worksheet.

Solution

$$20,000 = 5,000 a_{\overline{5}|i} \iff i = 7.9308\%$$

Example

An investor is asked to invest $1,000 and is promised payments of $350 in one year, $296 in two years, and $575 in three years. Find the IRR for this investment.

Example

An investor is asked to invest \$1,200 and is promised a return payment of \$550 in one year and \$730 in the second year. Find the internal rate of return.

Example

An investor is asked to invest \$1,300 and is promised a return payment of \$480 in one year, \$350 in two years, and \$650 in the three years. Find the internal rate of return.

Example

An investor is asked to invest \$12,000 and is promised return payments of \$3,500 for four years. Find the internal rate of return.

Example

An investor is asked to invest $25,000 and is promised 10 semiannual return payments of $3,000.

1. Find the internal rate of return compounded semiannually.
2. Find the effective annual return.

Remark

Question: why is the yield rate referred to as an "*internal* rate of return"?

Answer: the investment yields do not apply to money after it is paid out. If the investor re-invests the cash payments, this changes the overall return.

IRR *vs.* Reinvestment Problems

An investor is asked to invest \$1,000 and is promised payments of \$350 in one year, \$296 in two years, and \$575 in three years. The payments received are re-invested at an annual effective rate of 6%. Find the yield on this investment at the end of the third year.

Uniqueness of IRR

- The equation of value set up for the IRR over n time periods is a polynomial of degree n in the unknown v.
- A polynomial of degree n may have 0 to n real solutions.
- There is a unique real solution larger than -1 (or -100%) when $C_0 < 0$ and $C_k > 0$ for $k = 1, 2, \ldots, n$ (or when $C_0 > 0$ and $C_k < 0$ for $k = 1, 2, \ldots, n$).
- If we treat the cashflows as the period-by-period balances of a bank account and the account balances never change sign (negative to positive or positive to negative), the IRR is unique.

Example

An investment manager has a fund of $200,000 at the beginning of 2019. On February 1, the fund pays out $180,000. On March 1, the manager places an additional $120,000 in the fund. On April 1, the manager liquidates the fund and withdraws $200,000. What is the internal rate of return? Is it unique?

Solution

The IRR is the solution to the equation:

$$-200,000 + 180,000v - 120,000v^2 + 200,000v^3 = 0.$$

According to the CF worksheet, $i = 14.1671\%$. We can see if the IRR is unique by treating the investment as a savings account earning 14.1671% per month.

Date	Balance before CF	CF	Balance after CF
01/01/2019	0.00	200,000	200,000.00
02/01/2019	228,334.23	−180,000	48,334.23
03/01/2019	55,181.79	120,000	175,181.79
04/01/2019	200,000.00	−200,000	0.00

Since the balance is always non-negative the IRR is unique.

Example

An investor can invest \$200,000 in a mining operation. In one year she will receive a payment of \$460,000. In two years she must pay \$264,000 to close down and clean up the mine. Find the IRR.

Solution

$$
\begin{aligned}
-200000 + \frac{460000}{1+i} - \frac{264000}{(1+i)^2} &= 0 \\
i = 0.05 \quad or \quad i &= 0.10
\end{aligned}
$$

Both IRRs seem valid, but we should check using the savings account analysis.

Savings Account Analysis for IRR 5%

Year	Balance before CF	CF	Balance after CF
0	0	200, 000	200, 000
1	210, 000	−460, 000	−250, 000
2	−262, 500	264, 000	1, 500

At the end of year 1, the investor would have been in debt. The IRR is not used to analyze loans.

Savings Account Analysis for IRR 10%

Year	Balance before CF	CF	Balance after CF
0	0	$200,000$	$200,000$
1	$220,000$	$-460,000$	$-240,000$
2	$-264,000$	$264,000$	0

At the end of year 1, the investor would have been in debt. The IRR is not used to analyze loans.

Example

Pete buys 1500 shares of stock at \$6 per share and pays a commission to his broker of 2%. After six months he receives a dividend of \$0.25 per share which he immediately reinvests commission-free to buy shares at \$5 per share. Six months after that, Pete buys 1000 shares at \$5.50 per share and pays a 2% commission. In another six months he receives a dividend of \$0.20 per share and sells all his shares at \$6 per share and pays another 2% commission. Find the semi-annually compounded internal rate of return $i^{(2)}$.

Solution

Let the unit of time be six months.

t	C_t	**Shares Owned**
0	$-9,180$	1500
1	0	1575
2	$-5,610$	2575
3	$515 + 15,141$	0

$$\begin{aligned} -9180 - 5610(1+j)^{-2} + 15656(1+j)^{-3} &= 0 \\ j &= 0.025576 \end{aligned}$$

which implies $i^{(2)} = 5.1151\%$.

Savings Account Analysis

t	Balance before CF	CF	Balance after CF
0.0	0.00	9,180.00	9,180.00
0.5	9,414.79	0.00	9,414.79
1.0	9,655.58	5,610.00	15,265.58
1.5	15,656.01	−15,656.00	0.01

Since the savings account balance is always positive, the IRR is valid.

Dollar-Weighted Rate of Return (1 of 2)

Definition

The **dollar-weighted rate of return** is an internal rate of return calculated using simple interest from the time of transactions to the end of the year.

- A: initial balance in fund
- B: final balance in fund
- C: net total fund contributions,

$$C = \sum_{k=1}^{n} C_k.$$

- I: interest earned,

$$I = B - A - C.$$

Dollar-Weighted Rate of Return (2 of 2)

Let i be the (unknown) simple interest rate.

$$\begin{aligned}
B &= A(1+i) + \sum_{k=1}^{n} C_k(1 + i(1-t_k)) \\
B &= A + A\,i + i\sum_{k=1}^{n} C_k(1-t_k) + \sum_{k=1}^{n} C_k \\
B - A - C &= i\left(A + \sum_{k=1}^{n} C_k(1-t_k)\right) \\
i &= \frac{I}{A + \sum_{k=1}^{n} C_k(1-t_k)} = \frac{B - A - C}{A + \sum_{k=1}^{n} C_k(1-t_k)}
\end{aligned}$$

Example

An investment manager had a fund of $150,000 at the start of 2015. On April 1, 2015 a new deposit of $35,000 was made. On October 1, 2015 a withdrawal of $45,000 was made. At the end of 2015 the fund balance was again $150,000. Calculate the dollar-weighted rate of return.

Solution

$$\begin{aligned}
A &= 150,000 \\
B &= 150,000 \\
C &= 35,000 - 45,000 = -10,000 \\
I &= 150,000 - 150,000 - (-10,000) = 10,000 \\
i &= \frac{10,000}{150,000 + 35,000\left(1 - \frac{3}{12}\right) - 45,000\left(1 - \frac{9}{12}\right)} = 0.06061
\end{aligned}$$

Example

An investment manager had a fund of $200,000 at the start of the year. On May 1 the fund a had grown to $216,000 and a new deposit of $40,000 was made. On December 1 the fund balance was $260,000 and a withdrawal of $24,000 was made. At the end of the year the fund balance was $220,000. Calculate the dollar-weighted rate of return for the year.

Time-Weighted Rate of Return (1 of 2)

Definition

The **time-weighted rate of return** is found by compounding returns over successive parts of a year.

- A: initial balance in fund
- B: final balance in fund
- C_k: net fund contributions at time t_k
- F_k: fund balance immediately before the net fund contribution at time t_k

Time-Weighted Rate of Return (2 of 2)

The values of the fund immediately after time t_{k-1} and immediately before time t_k are respectively $F_{k-1} + C_{k-1}$ and F_k. Thus the interest rate for the interval $[t_{k-1}, t_k]$ is

$$\frac{F_k}{F_{k-1} + C_{k-1}} - 1.$$

The time-weighted return compounds these rates over the entire year.

$$i_T = \frac{F_1}{A} \cdot \frac{F_2}{F_1 + C_1} \cdot \frac{F_3}{F_2 + C_2} \cdots \frac{F_n}{F_{n-1} + C_{n-1}} \cdot \frac{B}{F_n + C_n} - 1$$

Example

An investment manager had a fund of $150,000 a the start of 2015. On April 1, 2015 the fund had grown to $162,000 and a new deposit of $35,000 was made. On October 1, 2015 the fund balance was $200,000 and a withdrawal of $45,000 was made. At the end of 2015 the fund balance was again $150,000. Calculate the time-weighted rate of return.

Solution

Note that $A = B = \$150,000$ and

i	F_i	C_i
1	$162,000$	$35,000$
2	$200,000$	$-45,000$

$$
\begin{aligned}
i_T &= \frac{162,000}{150,000} \cdot \frac{200,000}{162,000 + 35,000} \cdot \frac{150,000}{200,000 - 45,000} - 1 \\
&= 0.06108
\end{aligned}
$$

Example

An investment manager had a fund of \$200,000 at the start of the year. On May 1 the fund a had grown to \$216,000 and a new deposit of \$40,000 was made. On December 1 the fund balance was \$260,000 and a withdrawal of \$24,000 was made. At the end of the year the fund balance was \$220,000. Calculate the time-weighted rate of return for the year.

Example

A pension fund receives contributions and pays benefits from time to time. The fund value is reported after every transaction and at the end of the year. The details for 2018 were as shown.

	Date	Amount
Fund values:	01/01/2018	1,000,000
	03/01/2018	1,250,000
	09/01/2018	1,500,000
	11/01/2018	1,200,000
	01/01/2019	950,000
Contributions received:	02/28/2018	220,000
	08/31/2018	225,000
Benefits paid:	10/31/2018	500,000
	12/31/2018	250,000

Find the dollar-weighted and time-weighted rates of return for the pension fund.

Dollar-Weighted Return

$$
\begin{aligned}
I &= 950,000 - 1,000,000 - (220,000 + 225,000 - 500,000 - 250,000) \\
&= 255,000 \\
i &= \frac{255}{1,000 + 220(10/12) + 225(4/12) - 500(2/12) - 250(0)} \\
&= 0.2170
\end{aligned}
$$

Time-Weighted Return

Accumulation factors for portions of 2018:

$$01/01/2018 - 02/28/2018 \quad : \quad \frac{1,250,000 - 220,000}{1,000,000} = 1.03$$

$$03/01/2018 - 08/31/2018 \quad : \quad \frac{1,500,000 - 225,000}{1,250,000} = 1.02$$

$$09/01/2018 - 10/31/2018 \quad : \quad \frac{1,200,000 + 500,000}{1,500,000} = 1.13333$$

$$11/01/2018 - 12/31/2018 \quad : \quad \frac{950,000 + 250,000}{1,200,000} = 1$$

Thus the time-weighted return is

$$i_T = (1.03)(1.02)(1.13333)(1) - 1 = 0.1907.$$

Net Present Value

Definition

The **net present value** (**NPV**) of a series of cashflows C_0, C_1, ..., C_n at rate i is the sum of the present values of the cashflows.

$$NPV = C_0 + \frac{C_1}{1+i} + \frac{C_2}{(1+i)^2} + \cdots + \frac{C_n}{(1+i)^n}$$

If an investor has a interest rate preference, it can be used to rank the present values of sequences of cashflows.

- A sequence of cashflows with a negative net present value is not profitable.
- A sequence of cashflows with a positive net present value is profitable.

Example

Suppose an investor has cashflows of $C_0 = -2,000$, $C_1 = 1,200$, and $C_2 = 1,100$. Find the net present value of the cashflows when

- $i = 0.08$, ($NPV = 54.1838$)
- $i = 0.10$, ($NPV = 0.0000$)
- $i = 0.12$. ($NPV = -51.6582$)

Use the CF and NPV worksheets on the calculator.

Example

Suppose an investor wishes to investigate the sequence of cashflows $C_0 = -1.00$, $C_1 = 2.30$, and $C_2 = -1.32$.

1. Find the internal rate of return.
2. Find the net present value at $i = 15\%$.

IRR

The equation of value is

$$
\begin{aligned}
0 &= -1.00 + 2.30v - 1.32v^2 \\
v &= \frac{-2.30 \pm \sqrt{(2.30)^2 - 4(-1.32)(-1.00)}}{2(-1.32)} \\
v &= \frac{-2.30 \pm 0.10}{-2.64} \\
v = 0.833333 &\quad \text{or} \quad 0.909091 \\
i = 0.20 &\quad \text{or} \quad 0.10.
\end{aligned}
$$

Savings Account Analysis

Suppose $i = 10\%$.

t	Balance before CF	CF	Balance after CF
0	0.00	1.00	1.00
1	1.10	-2.30	-1.20
2	-1.32	1.32	0.00

Neither IRR is valid.

Suppose $i = 20\%$.

t	Balance before CF	CF	Balance after CF
0	0.00	1.00	1.00
1	1.20	-2.30	-1.10
2	-1.32	1.32	0.00

NPV

The NPV is

$$-1.00 + \frac{2.30}{1 + 0.15} - \frac{1.32}{(1 + 0.15)^2} = 0.00189.$$

Example

A bank is considering two alternative investments.

Investment A: requires an initial cashflow (out) of \$1M at $t = 0$ and will generate cashflows (in) of \$200K per year at the end of each year for 9 years.

Investment B: requires an initial cashflow (out) of \$1M at $t = 0$ and will generate cashflows (in) of \$300K per year at the end of each year for 5 years.

If the bank is able to borrow money at an annual effective rate of 10%, which investment has the higher net present value?

Solution

For investment A:

$$NPV = -1,000,000 + 200,000 a_{\overline{9}|0.10} = \$151,804.76.$$

For investment B:

$$NPV = -1,000,000 + 300,000 a_{\overline{5}|0.10} = \$137,236.03.$$

Remarks:

- Since Investment A has the higher NPV, it is the preferable investment.
- The same results could have been calculated using the CF and NPV worksheets on the calculator.

Portfolio Method

Money invested in some accounts may earn different rates of interest depending on how long the money has been invested.

- Money invested earns 4% the first year, 5% the second year, and 6% in years after.
- It is helpful to think of money in the account as segregated into "new money" (in the fund for less than two years) and "old money" (in the fund for more than two years).
- The term **portfolio year rate** refers to the rate earned by the old money, while **investment year rate** refers to the rate earned by the new money.

Table of Interest Rates

Calendar Year of Original Investment	Investment Year Rates (in %)					Portfolio Rates (in %)
y	i_1^y	i_2^y	i_3^y	i_4^y	i_5^y	i^{y+5}
2002	8.25	8.25	8.40	8.50	8.50	8.35
2003	8.50	8.70	8.75	8.90	9.00	8.60
2004	9.00	9.00	9.10	9.10	9.20	8.85
2005	9.00	9.10	9.20	9.30	9.40	9.10
2006	9.25	9.35	9.50	9.55	9.60	9.35
2007	9.50	9.50	9.60	9.70	9.70	
2008	10.00	10.00	9.90	9.80		
2009	10.00	9.80	9.70			
2010	9.50	9.50				
2011	9.00					

Reading the Table

- Money invested in year y, under the investment year method, earns interest specified by the values in row y.
- Money invested in year y, under the portfolio method, earns interest specified by the values in the Portfolio Rates column starting on row $y - 5$.

Example

Suppose $10,000 is invested on 01/01/2007.

1. Find the accumulated value on 01/01/2011 under the investment year method.
2. Find the accumulated value on 01/01/2011 under the portfolio method.
3. Find the accumulated value on 01/01/2011 if the balance is withdrawn at the end of every year and is reinvested at the new money rate.

Solution

Investment year method:

$$10,000(1.095)(1.095)(1.096)(1.097) = 14,416$$

Portfolio method:

$$10,000(1.0835)(1.086)(1.0885)(1.091) = 13,974$$

Reinvestment method:

$$10,000(1.095)(1.1)(1.1)(1.095) = 14,508$$

Homework

- Read Chapter 5
- Exercises: distributed on handout