Term Structure of Interest Rates

MATH 372 *Financial Mathematics I*

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Objectives

In this lesson we will learn:

- the relationship between interest rates and time to maturity of an investment,
- how to calculate implied rates of interest from the yield on bonds,
- how to calculate forward interest rates for loans to be taken in the future,
- the concept of arbitrage.

Intuitive Idea and Terminology

- Many factors influence the interest rates charged on loans.
  - **Credit rating** of the borrower,
  - length of time to maturity of the loan,
  - other economic factors.
- The relationship between the interest rate charged and the length of time to maturity of the loan is called the **yield curve**.

Yield Curve

The website treasury.gov keeps records of interest rates on US Treasury bonds of different maturities. For example the rates listed for March 26, 2019 were as follows.

U.S. Treasury Bonds	
Maturity	**Yield (in %)**
1 month	2.44
2 month	2.46
3 month	2.44
6 month	2.52
1 year	2.55
2 year	2.55
3 year	2.54
5 year	2.56
7 year	2.67
10 year	2.76
20 year	2.97
30 year	3.13

Yield Curve (03/26/2019)

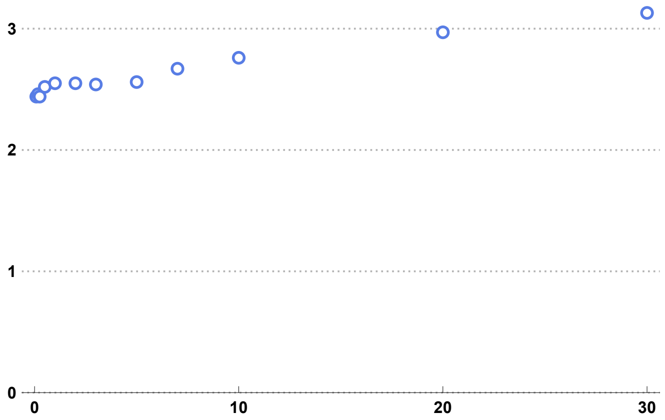
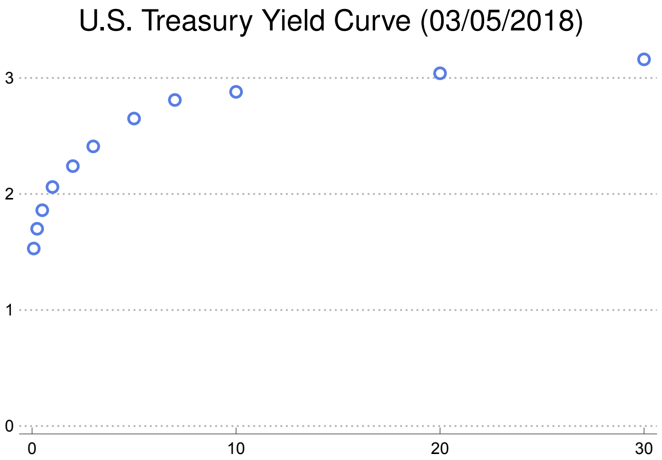
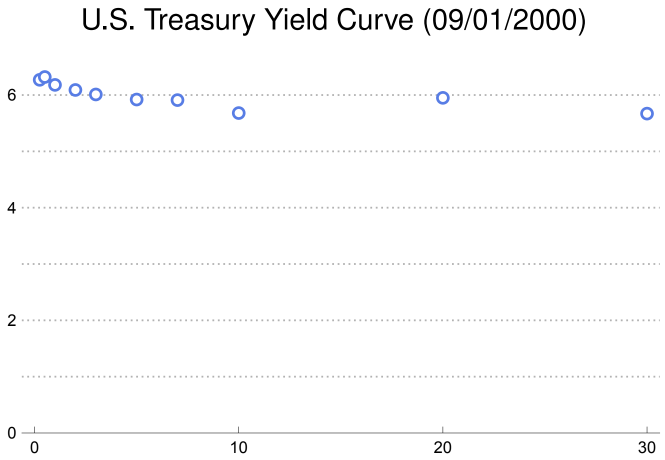


Illustration (Normal Yield Curve)

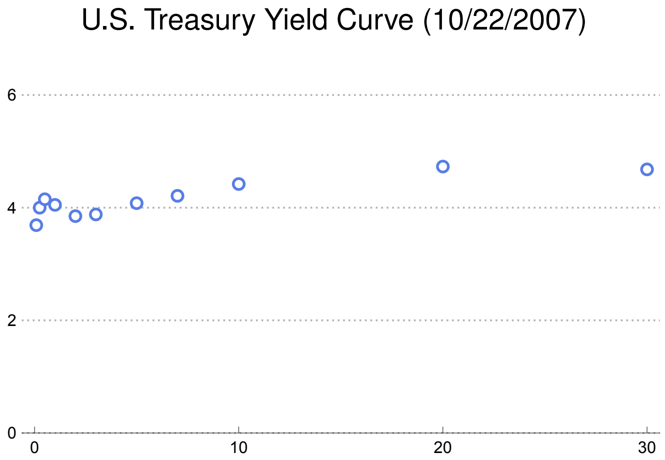


Generally the longer the term of the loan, the higher the interest rate.

Inverted Yield Curve



Humped Yield Curve



Zero Coupon Bonds

Definition

A **zero coupon bond** is a bond which has no coupons and has a single payment made at the time of maturity.

Remarks:

- A zero coupon bond can be called a **discount bond** or **pure discount bond**.
- A U.S. Treasury Bill is a zero coupon bond.
- The yield rates on U.S. Treasury bills can be used to determine the yield curve.

Treasury STRIPS

STRIPS: **S**eparate **T**rading of **R**egistered **I**nterest and **P**rincipal of **S**ecurities

- The US government does not issue zero bond bonds directly.
- Financial institutions purchase government coupon bonds and separately sell the coupons and redemption amounts to investors.

Risk

- The U.S. government is generally regarded as having the lowest risk of default of any borrower.
- Interest rates on short-term U.S. Treasury Bills are sometimes referred to as **risk-free rates**.
- The difference between another borrower's interest rate and the rate on a U.S. Treasury Bill will be called the **risk premium**.

Example

If a company's bonds have a yield of 5% and similar US government bonds have a yield of 3%, the difference $(5-3)\% = 2\%$ is extra compensation investors require for purchasing the company bonds.

Term Structure of Interest Rates

Definition

The **term structure of interest rates** at the current point in time is the set of yield rates on zero coupon bonds of all maturities. This set will be denoted $\{s_t\}_{t>0}$ where s_t is the annual effective yield rate as of time 0 for a zero coupon bond maturing at time t. The rate s_t is called the **spot rate.**

The price of a zero coupon bond with redemption value C maturing t years from now is

$$P = \frac{C}{(1+s_t)^t}.$$

Example

Suppose the current term structure has the following yields on zero coupon bonds.

Term	Zero Coupon Bond Rate
0.5 year	8.00%
1.0 year	9.00%
1.5 year	10.00%
2.0 year	11.00%

These bond rates are nominal, compounded semi-annually. Find the price of a $1,000 face amount and yield to maturity for each of the following 2-year bonds (with semi-annual coupons as appropriate).

1. A zero coupon bond.
2. A 4% annual coupon bond.
3. A 9% annual coupon bond.

Solution

1. Zero coupon bond:

$$\begin{aligned} P &= 1000\left(1+\frac{0.11}{2}\right)^{-4} = \$807.22 \\ \text{yield} &= 11\% \end{aligned}$$

2. 4% annual coupon bond:

$$\begin{aligned} P &= 20\left[(1.04)^{-1}+(1.045)^{-2}+(1.05)^{-3}\right]+1020(1.055)^{-4} \\ &= \$878.18 \\ 878.18 &= 20\,a_{\overline{4}|j}+1000v_j^4 \\ j &= 0.054733 \implies i^{(2)} = 10.9466\% \text{ (nominal)} \end{aligned}$$

3. 9% annual coupon bond:

$$\begin{aligned} P &= 45\left[(1.04)^{-1}+(1.045)^{-2}+(1.05)^{-3}\right]+1045(1.055)^{-4} \\ &= \$966.89 \\ 966.89 &= 45\,a_{\overline{4}|j}+1000v_j^4 \\ j &= 0.054434 \implies i^{(2)} = 10.8868\% \text{ (nominal)} \end{aligned}$$

Example

Term	Spot Rate
1	2.00%
2	3.00%
3	3.65%
4	4.10%
5	4.45%
6	4.75%
7	5.00%
8	5.20%

A four-year annual-coupon $1,000 par bond has a coupon rate of 3.5%. Use the yield curve to find:

1. the price of the bond,
2. the bond's yield to maturity.

Calculating the Spot Rate

The prices of coupon bonds can be used to determine the spot rates using a process called **bootstrapping**.

Suppose we know the following information about a sequence of coupon bonds.

Term	Coupon Rate	Price per 100 Face	Yield to Maturity
1	2.50%	100.00	2.500%
2	4.00%	101.76	3.079%
3	3.50%	100.54	3.308%
4	4.00%	101.81	3.517%
5	3.00%	97.19	3.625%

Calculate the spot rate for terms of 1 to 5 years.

Bootstrapping Process (1 of 4)

Year 1: the price of a bond which pays 102.50 in one year is 100.00, thus the 1-year spot rate is

$$s_1 = \frac{102.50}{100.00} - 1 = 0.0250 \implies 2.50\%.$$

Year 2: the bond pays 4 at the end of year 1 and 104 at the end of year 2. The present value of the first coupon is $\frac{4}{1.0250} = 3.9024$. The present value of the remaining coupon and redemption payment (104) is thus

$$101.76 - 3.9024 = 97.8576.$$

Therefore the 2-year spot rate solves

$$\frac{104}{(1+s_2)^2} = 97.8576 \iff s_2 = 0.0309 \implies 3.0907\%.$$

Bootstrapping Process (2 of 4)

Year 3: find the present value of the 3rd year coupon and redemption amount.

$$100.54 - \frac{3.50}{1.025} - \frac{3.50}{(1.030907)^2} = 93.8321$$

Therefore the 3-year spot rate is

$$\frac{103.50}{(1+s_3)^3} = 93.8321 \iff s_3 = 0.0332 \implies 3.3228\%.$$

Bootstrapping Process (3 of 4)

Year 4: find the present value of the 4th year coupon and redemption amount.

$$101.81 - \frac{4}{1.025} - \frac{4}{(1.030907)^2} - \frac{4}{(1.03228)^3} = 90.5174$$

Therefore the 4-year spot rate is

$$\frac{104}{(1+s_4)^4} = 90.5174 \iff s_4 = 0.0353 \implies 3.5322\%.$$

Bootstrapping Process (4 of 4)

Year 5: find the present value of the 5th year coupon and redemption amount.

$$97.19 - \frac{3}{1.025} - \frac{3}{(1.030907)^2} - \frac{3}{(1.03228)^3} - \frac{3}{(1.035322)^4} = 86.1095$$

Therefore the 5-year spot rate is

$$\frac{103}{(1+s_5)^5} = 86.1095 \iff s_5 = 0.0365 \implies 3.6471\%.$$

Summary

Term	Coupon Rate	Price per 100 Face	Yield to Maturity	Spot Rate
1	2.50%	100.00	2.500%	2.5000%
2	4.00%	101.76	3.079%	3.0907%
3	3.50%	100.54	3.308%	3.3228%
4	4.00%	101.81	3.517%	3.5322%
5	3.00%	97.19	3.625%	3.6471%

Example

Consider the yield curve table below.

Term	Spot Rate
1	5.00%
2	4.50%
3	4.00%
4	4.00%
5	4.00%

A 3-year annual-coupon $1,000 par value bond has a coupon rate of 4%. Use the yield curve table to find the price of the bond and the yield to maturity of the bond.

Forward Rates

Definition

The $(n-1)$**-year forward interest rate** is the interest rate agreed upon today for a 1-year loan to be made $n-1$ years from now. The $(n-1)$-year forward rate will be denoted $i_{n-1,n}$

Remarks:

- Example: $i_{2,3}$ is the interest rate for a one-year loan to be made 2 years from now.
- Example: $i_{0,1} = s_1$ the spot rate.
- The yield curve on zero coupon bonds are varying maturities implies the forward rates.
- Notation: $i_{m,m+n}$ is the interest rate for an n-year loan to begin m years from now.

Implied Forward Rate

Given the term structure of zero coupon bond yield rates $\{s_t\}_{t>0}$, the $(n-1)$-year forward rate for the year $[n-1, n]$ is given in the equation

$$1 + i_{n-1,n} = \frac{(1+s_n)^n}{(1+s_{n-1})^{n-1}}.$$

The equation above is equivalent to the following equation.

$$(1+s_n)^n = (1+i_{n-1,n})(1+s_{n-1})^{n-1}$$

The left-hand and right-hand sides represent two ways to make an n-year investment.

Spot Rates and Forward Rates

$$\begin{aligned}(1+s_n)^n &= (1+i_{n-1,n})(1+s_{n-1})^{n-1}\\ &= (1+i_{n-1,n})(1+i_{n-2,n-1})(1+s_{n-2})^{n-2}\\ &\vdots\\ &= (1+i_{n-1,n})(1+i_{n-2,n-1})\cdots(1+i_{0,1})\end{aligned}$$

Example

Suppose the yield curve is based on the values in the following table.

Year	Spot Rate
1	5%
2	6%
3	7%
4	8%

Find the 3-year forward rate.

Solution

$$\begin{aligned} 1 + i_{3,4} &= \frac{(1+0.08)^4}{(1+0.07)^3} \\ i_{3,4} &= 0.1106 \implies i_{3,4} = 11.06\% \end{aligned}$$

Example

Given the spot rates in the table below, calculate the corresponding forward rates $i_{n-1,n}$ for $n = 1, 2, 3, 4, 5$.

n	s_n
1	5.0%
2	6.2%
3	6.8%
4	7.3%
5	7.7%

Example

Suppose the forward rates for 1-year loans are given in the following table.

Year	Forward Rate
0	3.5%
1	4.2%
2	4.5%
3	5.2%

Find the 4-year spot rate.

Solution

$$
\begin{aligned}
(1+s_4)^4 &= (1+0.035)(1+0.042)(1+0.045)(1+0.052) \\
s_4 &= 0.0435 \implies s_4 = 4.35\%
\end{aligned}
$$

Example

Given the forward rates in the table below, calculate the corresponding spot rates s_n for $n = 1, 2, 3, 4, 5$.

n	$i_{n-1,n}$
1	5.0%
2	6.2%
3	6.8%
4	7.3%
5	7.7%

Example

Consider the yield curve table below.

Term	**Spot Rate**
1	5.00%
2	4.50%
3	4.00%
4	4.00%
5	4.00%

Find the 1-year and 2-year forward rates.

Example

Suppose the yield curve is given by the function

$$s_k = 0.075 + 0.005k - 0.0015k^2$$

where k is years to maturity.

1. Calculate the price of a 3-year \$1,000 par bond with 8% annual coupons.
2. Find the 3-year forward rate.

Solution

1. Price of bond:

$$
\begin{aligned}
s_1 &= 0.075 + 0.005(1) - 0.0015(1^2) = 0.0785 \\
s_2 &= 0.075 + 0.005(2) - 0.0015(2^2) = 0.0790 \\
s_3 &= 0.075 + 0.005(3) - 0.0015(3^2) = 0.0765 \\
P &= \frac{80}{1.0785} + \frac{80}{(1.0790)^2} + \frac{1080}{(1.0765)^3} = \$1008.62
\end{aligned}
$$

2. 3-year forward rate:

$$
\begin{aligned}
s_4 &= 0.075 + 0.005(4) - 0.0015(4^2) = 0.0710 \\
i_{3,4} &= \frac{(1+s_4)^4}{(1+s_3)^3} - 1 = \frac{1.3157}{1.2475} - 1 = 0.054668
\end{aligned}
$$

Example

Fill in the missing entries in the table below. All interest rates are annual effective rates.

n	**Spot Rate** s_n	**Forward Rate** $i_{n-1,n}$	**Accumulation Factor** $a(n)$
1	3.40%	3.40%	1.0340
2		4.40%	1.0795
3	4.30%		
4			1.1994
5			1.2732
6		6.45%	

Arbitrage

Definition

An **arbitrage** results from the simultaneous purchase and sale of securities in different markets in order to profit from price discrepancies.

Remarks:

- With arbitrage a financial transaction returns a positive profit on an investment of 0 with no risk of loss.
- In arbitrage a risk-free investment has a guaranteed return greater than the risk-free rate of return.
- An arbitrage is a “free lunch”.

Real-life Arbitrage (1 of 2)

2006 Winter Olympics, Turin, Italy

- Online casino, SportingUSA.com (now out of business) offerred $2.5 : 1$ odds against Denmark winning medals (this implies the probability of Denmark winning no medals is $p = 0.2857$).
- Online casino Bet365.com (still in business) offerred $1.875 : 1$ odds Denmark would win at least one medal (this implies the probability of Denmark winning at least one medal is $q = 0.6522 \neq 1 - p$).
- Suppose a bettor had \$1,000 to bet and wagered \$500 at each casino.
- Is it possible to guarantee a positive profit?

Real-life Arbitrage (2 of 2)

- If Denmark does not win a medal the bettor receives

$$(500)(2.5 + 1) = \$1,750.$$

- If Denmark wins at least one medal the bettor receives

$$(500)(1.875 + 1) = \$1,437.50$$

- In the worst case, the bettor has invested \$1,000 and received $\$1,437.50$.
- Denmark did not medal in 2006.

Forward Rate Arbitrage

We have used the yield curve to determine the implied forward rates for loans to be made in the future. Any forward rate differing from the implied one, represents the existence of arbitrage.

Example

Suppose the 1-year zero coupon bond yield rate is 7% and the 2-year zero coupon bond rate is 8% (effective annual rates). Suppose it is possible to borrow or lend at the zero coupon rates during their maturity periods.

1. What is the 1-year forward rate?
2. If a borrower offers to pay you an interest rate greater than the 1-year forward rate for a loan to be made one year from now, how can you turn an arbitrage profit?
3. If a lender offers you a loan at an interest rate lower than the 1-year forward rate for a loan to be made one year from now, how can you turn an arbitrage profit?

Solution

1. 1-year forward rate:

$$i_{1,2} = \frac{(1+0.08)^2}{1+0.07} - 1 = 0.0901$$

2. If the borrower offers to pay interest at rate $i > 9.01\%$:
   - Borrow P for 2 years at 8%.
   - Invest P for 1 year at 7%.
   - Lend $(1.07)P$ for 1 year at i.
   - Receive $(1+i)(1.07)P$ and payback loan $(1.08)^2P$.

$$\text{Profit} = (1+i)(1.07)P - (1.08)^2P > 0$$

3. If lender offers loan at interest rate $i < 9.01\%$:
   - Borrow P for 1 year at 7%.
   - Invest P for 2 years at 8%.
   - Borrow $(1.07)P$ from lender one year from now to pay back first loan.
   - At end of second year receive $(1.08)^2P$ and pay back second loan $(1+i)(1.07)P$.

$$\text{Profit} = (1.08)^2P - (1+i)(1.07)P > 0$$

Homework

- Read Chapter 6
- Exercises: distributed on handout