Differentiation and Integration of Fourier Series

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn about

- the properties of the derivatives of Fourier series,
- the properties of the integrals of Fourier series, and
- Parseval's Identity and Bessel's Inequality.

Cautionary Example

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$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$$

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2. Differentiate the series term by term.

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\cos(nx)$$

3. Does the result converge to $f'(x) = 1$? No, in fact the result diverges for all $x \in (-\pi, \pi)$.

Differentiation of Fourier Series

Theorem

Assume that $f(x)$ is continuous on $(-L, L)$ with $f(-L+) = f(L-)$, and $f'(x)$ is piecewise continuous on $(-L, L)$. Then the Fourier series of $f(x)$ can be differentiated term-by-term.

Example

- The Fourier (cosine) series for the 2π-periodic extension of $F(x) = x^2$ is

$$F(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

- The Fourier (sine) series for the 2π-periodic extension of $f(x) = 2x$ is

$$f(x) = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

- **Note**: the term-by-term derivative of the Fourier series for $F(x)$ results in the Fourier series for $f(x) = F'(x)$.

Proof (1 of 2)

- Suppose $f(x)$ is continuous on $(-L, L)$ and $f(-L+) = f(L-)$.
- Define $f(L) = f(L-) = f(-L+)$ which will make the $2L$-periodic extension of $f(x)$ continuous on $(-\infty, \infty)$.
- The Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

- If $f'(x)$ is piecewise continuous then $f'(x)$ has a Fourier series representation and

$$f'(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{n\pi x}{L} + \beta_n \sin \frac{n\pi x}{L} \right).$$

except at the removable or jump discontinuities of $f'(x)$.

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  except at the removable or jump discontinuities of $f'(x)$.
- We need to show $\alpha_0 = 0$ and

$$\alpha_n = \frac{n\pi}{L} b_n \quad \text{and} \quad \beta_n = -\frac{n\pi}{L} a_n.$$

Proof (2 of 2)

$$\alpha_0 = \frac{1}{L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{L}\left[f(x)\right]_{-L}^{L} = 0$$

$$\begin{aligned}\alpha_n &= \frac{1}{L}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,dx \\ &= \frac{1}{L}\left[f(x)\cos\frac{n\pi x}{L}\right]_{-L}^{L} + \frac{n\pi}{L}\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{n\pi}{L} b_n\end{aligned}$$

Using integration by parts. Similarly it is shown that $\beta_n = -\frac{n\pi}{L} a_n$.

Consequence of Differentiation Theorem

The assumptions of the theorem include the continuity of the $2L$-periodic extension of $f(x)$ and the piecewise continuity of the $2L$-periodic extension of $f'(x)$. Thus the Dirichlet Convergence Theorem states

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

for all $x \in (-\infty, \infty)$. The Fourier series for $f'(x)$ is

$$f'(x) \sim \frac{\pi}{L} \sum_{n=1}^{\infty} \left(n\, b_n \cos \frac{n\pi x}{L} - n\, a_n \sin \frac{n\pi x}{L} \right),$$

and for any $(-\infty, \infty)$

$$\frac{f'(x+) + f'(x-)}{2} = \frac{\pi}{L} \sum_{n=1}^{\infty} \left(n\, b_n \cos \frac{n\pi x}{L} - n\, a_n \sin \frac{n\pi x}{L} \right)$$

Integration of Fourier Series

Theorem

Let $f(x)$ be piecewise continuous on $[-L, L]$. The Fourier series of $f(x)$ can be integrated term by term and the resulting series always converges to the integral of $f(x)$ on $[-L, L]$. That is, if

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

then

$$\int_0^x f(s)\, ds = \frac{a_0}{2} x + \frac{L}{\pi} \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin \frac{n\pi x}{L} + \frac{b_n}{n} \left(1 - \cos \frac{n\pi x}{L} \right) \right].$$

Proof (1 of 4)

- Define $F(x) = \int_0^x f(s)\,ds$ and define $G(x) = F(x) - a_0 x/2$.
- Since $f'(x)$ is piecewise continuous on $[-L, L]$ then $F(x)$ and $G(x)$ are continuous on $(-L, L)$.
- By definition

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(s)\,ds$$

$$\frac{a_0 L}{2} + \frac{a_0 L}{2} = -\int_0^{-L} f(s)\,ds + \int_0^{L} f(s)\,ds$$

$$\int_0^{-L} f(s)\,ds + \frac{a_0 L}{2} = G(-L) = G(L) = \int_0^{L} f(s)\,ds - \frac{a_0 L}{2}.$$

- Therefore $G(x)$ has a Fourier series

$$G(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right).$$

Proof (2 of 4)

The Fourier coefficients for $G(x)$ are by definition,

$$\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{L} G(x)\cos\frac{n\pi x}{L}\,dx \\
&= \frac{1}{n\pi}\left(\left[\left(G(x)\sin\frac{n\pi x}{L}\right)\right]_{-L}^{L} - \int_{-L}^{L}(f(x)-\frac{a_0}{2})\sin\frac{n\pi x}{L}\,dx\right) \\
&= -\frac{1}{n\pi}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{a_0}{2n\pi}\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx \\
&= -\frac{L}{n\pi}b_n
\end{aligned}$$

using integration by parts. Similarly it can be shown that $B_n = \dfrac{L}{n\pi}a_n$.

Proof (3 of 4)

Thus far we have shown

$$G(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(-\frac{L}{n\pi} b_n \cos \frac{n\pi x}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right)$$

Proof (3 of 4)

Thus far we have shown

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G(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(-\frac{L}{n\pi} b_n \cos \frac{n\pi x}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right) \\
G(0) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(-\frac{L}{n\pi} b_n \cos \frac{n\pi (0)}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi (0)}{L} \right) \\
0 &= \frac{A_0}{2} - \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n \\
\frac{A_0}{2} &= \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n.
\end{aligned}$$

Proof (4 of 4)

Consequently

$$G(x) = \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n + \sum_{n=1}^{\infty} \left(-\frac{L}{n\pi} b_n \cos \frac{n\pi x}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right)$$

$$= \sum_{n=1}^{\infty} \frac{L}{n\pi} \left(b_n \left(1 - \cos \frac{n\pi x}{L} \right) + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right)$$

$$F(x) = \frac{a_0}{2} x + \frac{L}{\pi} \sum_{n=1}^{\infty} \left(\frac{b_n}{n} \left(1 - \cos \frac{n\pi x}{L} \right) + \frac{a_n}{n} \sin \frac{n\pi x}{L} \right).$$

Partial Sum of Fourier Series

Let the Nth partial sum of the Fourier series for $f(x)$ be

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

- We wish to describe the error present in this approximation.
- Multiply S_N by $f(x)$ and integrate over $[-L, L]$.
- Multiply S_N by $S_N(x)$ and integrate over $[-L, L]$.

Multiplication by $f(x)$

$$
\begin{aligned}
&\int_{-L}^{L} f(x) S_N(x)\, dx \\
&= \frac{1}{2} a_0 \int_{-L}^{L} f(x)\, dx \\
&\quad + \sum_{n=1}^{N} \left(a_n \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx + b_n \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \right) \\
&= \frac{L}{2} a_0^2 + L \sum_{n=1}^{N} \left(a_n^2 + b_n^2 \right) .
\end{aligned}
$$

Multiplication by $S_N(x)$

$$\int_{-L}^{L} S_N(x) S_N(x)\,dx = \frac{1}{2} a_0 \int_{-L}^{L} S_N(x)\,dx$$
$$+ \sum_{n=1}^{N} \left(a_n \int_{-L}^{L} S_N(x) \cos \frac{n\pi x}{L}\,dx + b_n \int_{-L}^{L} S_N(x) \sin \frac{n\pi x}{L}\,dx \right)$$
$$= \frac{L}{2} a_0^2 + L \sum_{n=1}^{N} \left(a_n^2 + b_n^2 \right).$$

This is the same result as before.

Squared Error

One measure of the error in approximating $f(x)$ by $S_N(x)$ is

$$0 \leq \int_{-L}^{L} (f(x) - S_N(x))^2 \, dx$$

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$$\begin{aligned} 0 &\leq \int_{-L}^{L} (f(x) - S_N(x))^2 \, dx \\ &= \int_{-L}^{L} (f(x))^2 \, dx - 2\int_{-L}^{L} f(x) S_N(x) \, dx \\ &\quad + \int_{-L}^{L} (S_N(x))^2 \, dx \\ &= \int_{-L}^{L} (f(x))^2 \, dx - \left(\frac{L}{2} a_0^2 + L \sum_{n=1}^{N} (a_n^2 + b_n^2) \right) \end{aligned}$$

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$$\frac{1}{2} a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^{L} (f(x))^2 \, dx.$$

This is known as **Bessel's inequality**.

Remarks

$$\sum_{n=1}^{N} (a_n^2 + b_n^2) \le \frac{1}{L} \int_{-L}^{L} (f(x))^2 \, dx$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} (a_n^2 + b_n^2) \le \frac{1}{L} \int_{-L}^{L} (f(x))^2 \, dx$$

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) \le \frac{1}{L} \int_{-L}^{L} (f(x))^2 \, dx$$

- A function for which $\int_{-L}^{L} (f(x))^2 \, dx$ is finite is said to be **square integrable on** $[-L, L]$.

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- A function for which $\int_{-L}^{L} (f(x))^2 \, dx$ is finite is said to be **square integrable on** $[-L, L]$.
- If $f(x)$ is square integrable on $[-L, L]$ then the infinite series $\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$ converges.

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- A function for which $\int_{-L}^{L} (f(x))^2 \, dx$ is finite is said to be **square integrable on** $[-L, L]$.
- If $f(x)$ is square integrable on $[-L, L]$ then the infinite series $\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$ converges.
- The convergence of the infinite series implies $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$.

Mean Square Error

For $N \in \mathbb{N}$ define the **mean square error** of the partial sum $S_N(x)$ relative to $f(x)$ to be

$$E_N = \frac{1}{2L} \int_{-L}^{L} (f(x) - S_N(x))^2 \, dx.$$

Theorem

Assume that $f(x)$ is a square integrable function on $[-L, L]$, then

$$\lim_{N \to \infty} E_N = 0.$$

Corollary

Corollary

If $f(x)$ is a square integrable function on $[-L, L]$, then

$$\frac{1}{2L}\int_{-L}^{L}(f(x))^2\,dx = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

where a_n and b_n are the Fourier coefficients of $f(x)$.

Remark: this equation is known as **Parseval's identity** and functions like the Pythagorean theorem for Fourier series.

Application of Parseval's Identity

Find the $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$ using

- the Fourier (sine) coefficients for $f(x) = x^3$ on $(-\pi, \pi)$,
- Parseval's identity,
- and the facts that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Solution

Fourier coefficients:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin(nx)\,dx = \frac{2(-1)^n(6-n^2\pi^2)}{n^3}$$

Parseval's identity:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(x^3)^2\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{2(-1)^n(6-n^2\pi^2)}{n^3}\right]^2$$

$$\frac{\pi^6}{7} = \sum_{n=1}^{\infty}\frac{72}{n^6} - \sum_{n=1}^{\infty}\frac{24\pi^2}{n^4} + \sum_{n=1}^{\infty}\frac{2\pi^4}{n^2}$$

$$\frac{\pi^6}{7} + \frac{4\pi^6}{15} - \frac{\pi^6}{3} = 72\sum_{n=1}^{\infty}\frac{1}{n^6}$$

$$\frac{\pi^6}{945} = \sum_{n=1}^{\infty}\frac{1}{n^6}$$

Proof of Parseval's Identity

By definition

$$\begin{aligned}
E_N &= \frac{1}{2L}\int_{-L}^{L}(f(x)-S_N(x))^2\,dx \\
&= \frac{1}{2L}\int_{-L}^{L}(f(x))^2\,dx - \frac{1}{L}\int_{-L}^{L}f(x)S_N(x)\,dx + \frac{1}{2L}\int_{-L}^{L}(S_N(x))^2\,dx \\
&= \frac{1}{2L}\int_{-L}^{L}(f(x))^2\,dx - \left[\frac{1}{2}a_0^2 + \sum_{n=1}^{N}(a_n^2+b_n^2)\right] + \left[\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{N}(a_n^2+b_n^2)\right] \\
&= \frac{1}{2L}\int_{-L}^{L}(f(x))^2\,dx - \frac{1}{4}a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2+b_n^2).
\end{aligned}$$

Since $\lim_{N\to\infty} E_N = 0$ then

$$\frac{1}{2L}\int_{-L}^{L}(f(x))^2\,dx = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{N}(a_n^2+b_n^2).$$

Homework

- Read Sections 3.8 and 3.9
- Exercises: 20–25