

# Applications of First-Order PDEs

MATH 467 *Partial Differential Equations*

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# Objectives

In this lesson we will apply the solution techniques learned earlier to finding the solutions to some first-order PDEs arising in the natural and physical sciences.

# Structured Population Growth

- ▶ Consider a population consisting of individuals of different ages or sizes  $x$  at time  $t$ .
- ▶ Define  $p(x, t)$  to be age-density of the population. The number of individuals in the population between ages  $x$  and  $x + \Delta x$  at time  $t$  is approximately

$$p(x, t) \Delta x.$$

- ▶ Define  $N(t)$  to be the total population at time  $t$ ,

$$N(t) = \int_0^{\infty} p(x, t) dx.$$

We wish to develop a PDE model for the evolution of the age-density.

# Calendar Time and Age

- ▶ Population with ages in  $[x, x + \Delta x]$  at time  $t + \Delta t$  is approximately  $p(x, t + \Delta t) \Delta x$ .
- ▶ If individuals age at the same rate as time passes then

$$p(x, t + \Delta t) \Delta x = p(x - \Delta t, t) \Delta x$$

in other words, if deaths are ignored the number of individuals with ages  $[x, x + \Delta x]$  at time  $t + \Delta t$  is the same as the number of individuals with ages  $[x - \Delta t, x]$  at time  $t$ .

- ▶ Let  $\mu(x)$  the rate of death (deaths per unit time) of individuals of age  $x$ . The number of deaths in the population of individuals with ages  $[x - \Delta t, x]$  between times  $t$  and  $t + \Delta t$  is approximately

$$\mu(x - \Delta t)p(x - \Delta t, t) \Delta x \Delta t.$$

# Population Equation

$$\rho(x, t + \Delta t) \Delta x \approx \rho(x - \Delta t, t) \Delta x - \mu(x - \Delta t) \rho(x - \Delta t, t) \Delta x \Delta t$$

$$\rho(x, t + \Delta t) - \rho(x - \Delta t, t) \approx -\mu(x - \Delta t) \rho(x - \Delta t, t) \Delta t$$

Add and subtract  $\rho(x, t)$  on the left-hand side of the approximation.

$$\rho(x, t + \Delta t) - \rho(x, t) + \rho(x, t) - \rho(x - \Delta t, t) \approx -\mu(x - \Delta t) \rho(x - \Delta t, t) \Delta t$$

$$\frac{\rho(x, t + \Delta t) - \rho(x, t)}{\Delta t} + \frac{\rho(x, t) - \rho(x - \Delta t, t)}{\Delta t} \approx -\mu(x - \Delta t) \rho(x - \Delta t, t)$$

$$\rho_t + \rho_x = -\mu(x) \rho$$

# von Foerster Equation

We have the first-order linear PDE

$$p_t + p_x = -\mu(x) p$$

with side condition

$$p(x, 0) = f(x),$$

specifying the initial distribution of the ages within the population, and

$$p(0, t) = \phi(t)$$

the density of new births at time  $t$ .

## Solution (1 of 2)

- ▶ Using the method of characteristics,

$$\frac{dx}{dt} = 1 \implies x - t = k$$

where  $k$  is a constant.

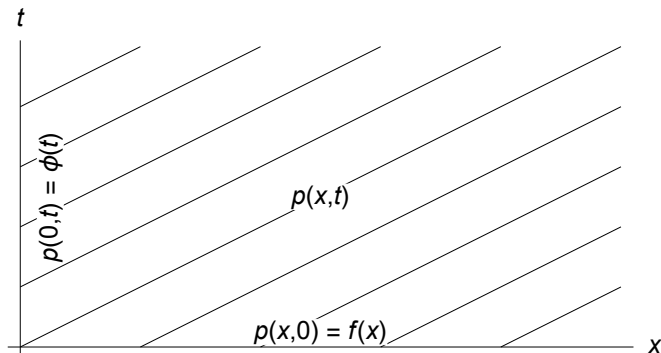
- ▶ Assuming  $p(x, t) = p(x(t), t) = p(t)$  then

$$\frac{dp}{dt} = p_t + p_x \frac{dx}{dt} = p_t + p_x = -\mu(x)p = -\mu(t+k)p.$$

- ▶ Solving this first-order linear ODE yields,

$$p(t) = C(k)e^{-\int_{t_0}^t \mu(s+k) ds}.$$

# Characteristics



$$p(t) = C(k)e^{-\int_{t_0}^t \mu(s+k) ds}.$$

Function  $p(t)$  is defined for  $t \geq 0$  when  $k \geq 0$  and is defined for  $t \geq -k$  when  $k < 0$ .

## Solution (2 of 2)

Apply side conditions.

$$p(0) = C(k) = \begin{cases} f(k) & \text{if } k \geq 0, \\ \phi(-k) & \text{if } k < 0. \end{cases}$$

Thus the solution can be expressed as

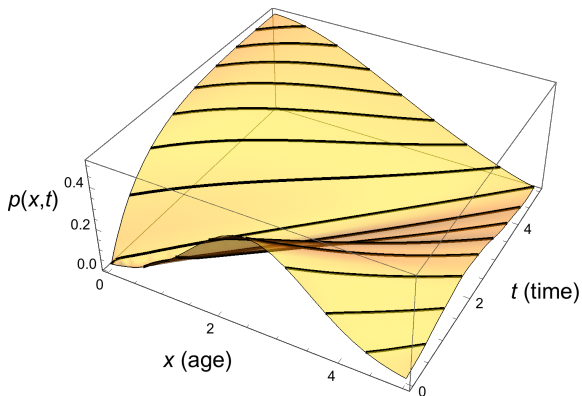
$$\begin{aligned} p(x, t) &= \begin{cases} f(x-t)e^{\int_0^t \mu(s+x-t) ds} & \text{if } k \geq 0, \\ \phi(t-x)e^{-\int_{-k}^t \mu(s+x-t) ds} & \text{if } k < 0. \end{cases} \\ &= \begin{cases} f(x-t)e^{-\int_{x-t}^x \mu(s) ds} & \text{if } t < x, \\ \phi(t-x)e^{-\int_0^x \mu(s) dx} & \text{if } t > x. \end{cases} \end{aligned}$$

# Example

$$\phi(t) = \frac{e^{-25/8}}{\sqrt{2\pi}} + \frac{1}{2}(1 - e^{-t})$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-5/2)^2/2}$$

$$\mu(x) = 1 - e^{-x/5}$$



# Comments

A reasonable assumption is that new births at time  $t$  depend on the distribution of ages in the population and/or the total population.

If the birth rate is age-dependent denote it as  $\beta(x)$  and then,

$$p(0, t) = \int_0^{\infty} \beta(x)p(x, t) dx.$$

This is called as **nonlocal boundary condition** since  $p(0, t)$  depends on the unknown density  $p(x, t)$ .

# Nonlocal Boundary Condition

$$\begin{aligned}\phi(t) &= p(0, t) = \int_0^{\infty} \beta(x)p(x, t) dx \\ &= \int_0^t \beta(x)p(x, t) dx + \int_t^{\infty} \beta(x)p(x, t) dx \\ &= \int_0^t \beta(x)\phi(t-x)e^{-\int_0^x \mu(s) ds} dx + \int_t^{\infty} \beta(x)f(x-t)e^{-\int_{x-t}^x \mu(s) ds} dx\end{aligned}$$

# Volterra Integral Equation

Define  $\psi(t) = \int_t^\infty \beta(x)f(x-t)e^{-\int_{x-t}^x \mu(s) ds} dx$ , then

$$\phi(t) = \int_0^t \beta(x)\phi(t-x)e^{-\int_0^x \mu(s) ds} dx + \psi(t)$$

which is an example of a **Volterra integral equation**.

If the integral equation can be solved for  $\phi(t)$  then, as before,

$$p(x, t) = \begin{cases} f(x-t)e^{-\int_{x-t}^x \mu(s) ds} & \text{if } t < x, \\ \phi(t-x)e^{-\int_0^x \mu(s) ds} & \text{if } t > x. \end{cases}$$

# Traffic Flow

Consider a one-way, single lane road and the vehicles on it.

- ▶ Vehicles are not allowed to pass each other.
- ▶ The road has no exits or entrances. All vehicles on the road are already present. No additional vehicles will be added and none may leave.
- ▶ Function  $\rho(x, t)$  denotes the density of vehicles, the number of vehicles per unit length of road (for example, cars/mile) at time  $t$  at the location  $x$ .
- ▶ Function  $q(x, t)$  denotes the flow of traffic, the number of vehicles passing location  $x$  and at time  $t$ .

## Model Development (1 of 3)

In consider the interval  $[a, b]$  along the road. The total number of vehicles in interval  $[a, b]$  given by

$$N(t) = \int_a^b \rho(x, t) dx.$$

The time rate of change of  $N(t)$  is

$$\frac{dN}{dt} = \frac{d}{dt} \left[ \int_a^b \rho(x, t) dx \right] = q(a, t) - q(b, t).$$

**Question:** what is an intuitive interpretation of the last equation?

## Model Development (2 of 3)

$$\begin{aligned} q(a, t) - q(b, t) &= - \int_a^b \frac{\partial}{\partial x} [q(x, t)] dx \\ &= \frac{d}{dt} \left[ \int_a^b \rho(x, t) dx \right] \end{aligned}$$

Equating the right-hand sides and re-arranging terms produces:

$$\int_a^b \left( \frac{\partial}{\partial t} [\rho(x, t)] + \frac{\partial}{\partial x} [q(x, t)] \right) dx = 0,$$

which implies (since  $[a, b]$  is arbitrary),

$$\rho_t + q_x = 0.$$

## Model Development (3 of 3)

The PDE:

$$\rho_t + q_x = 0$$

contains 2 unknown functions. If we assume  $q(x, t) = \rho(x, t)u(\rho(x, t))$  where  $u$  is the velocity function for a vehicle, then we have

$$\rho_t + (\rho u)_x = 0$$

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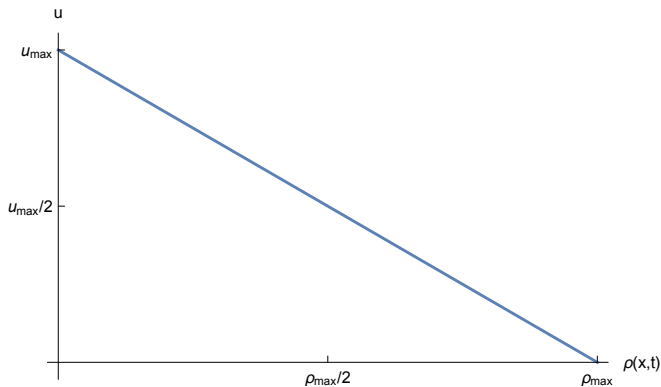
which contains only the unknown traffic density.

A common choice for  $u$  is

$$u = u_{\max} \left( 1 - \frac{\rho(x, t)}{\rho_{\max}} \right).$$

# Vehicle Speed vs. Density

$$u = u_{\max} \left( 1 - \frac{\rho(x, t)}{\rho_{\max}} \right).$$



# Quasilinear PDE

Using the common choice for  $u$  then

$$\begin{aligned}\rho_t + (\rho u)_x &= 0 \\ \rho_t + u_{\max} \left[ \rho \left( 1 - \frac{\rho}{\rho_{\max}} \right) \right]_x &= 0 \\ \rho_t + u_{\max} \left( 1 - \frac{2\rho}{\rho_{\max}} \right) \rho_x &= 0,\end{aligned}$$

which is a first-order quasilinear PDE.

We can solve this PDE with side condition  $\rho(x, 0) = f(x)$ .

# Characteristic System

$$\frac{dt}{d\tau} = 1$$

$$\frac{dx}{d\tau} = u_{\max} \left( 1 - \frac{2\rho}{\rho_{\max}} \right)$$

$$\frac{d\rho}{d\tau} = 0$$

# Characteristic System

$$\frac{dt}{d\tau} = 1 \implies t(\tau) = \tau + A$$

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$$\frac{dt}{d\tau} = 1 \implies t(\tau) = \tau + A$$

$$\frac{dx}{d\tau} = u_{\max} \left( 1 - \frac{2\rho}{\rho_{\max}} \right) \implies x(\tau) = u_{\max} \left( 1 - \frac{2B}{\rho_{\max}} \right) \tau + C$$

$$\frac{d\rho}{d\tau} = 0 \implies \rho(\tau) = B$$

where  $A$ ,  $B$ , and  $C$  are constants.

# Characteristic Curves

- ▶ The initial density defined a non-characteristic curve  $(0, s, f(s))$ .
- ▶ Suppose the characteristics intersect this curve at  $\tau = 0$ .

$$t(0) = 0 = A$$

$$x(0) = s = C$$

$$\rho(0) = f(s) = B$$

which implies

$$t = \tau$$

$$x = u_{\max} \left( 1 - \frac{2f(s)}{\rho_{\max}} \right) \tau + s$$

$$\rho = f(s).$$

## Elimination of $\tau$

Since  $t = \tau$ ,

$$x = u_{\max} \left( 1 - \frac{2f(s)}{\rho_{\max}} \right) t + s$$
$$\rho = f(s).$$

Consider the characteristics in the  $xt$ -plane.

- ▶ The characteristics are straight lines.
- ▶ The  $x$ -intercept lies at  $(s, 0)$ .
- ▶ The density of vehicles is constant (with value  $f(s)$ ) along this characteristic.
- ▶ If  $f(s) = f_0$  a constant, then all characteristics are parallel.

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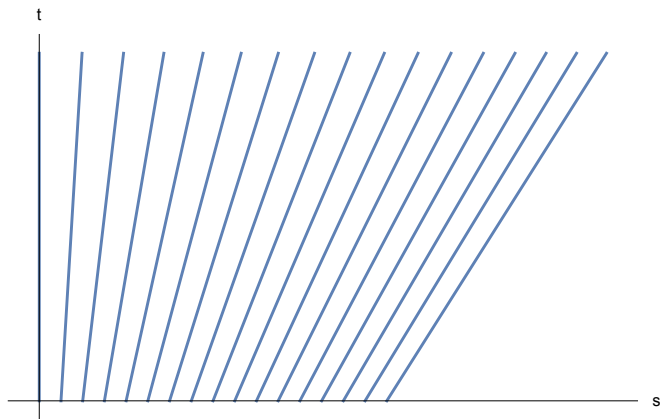
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**Question:** what if  $f(s)$  is not constant?

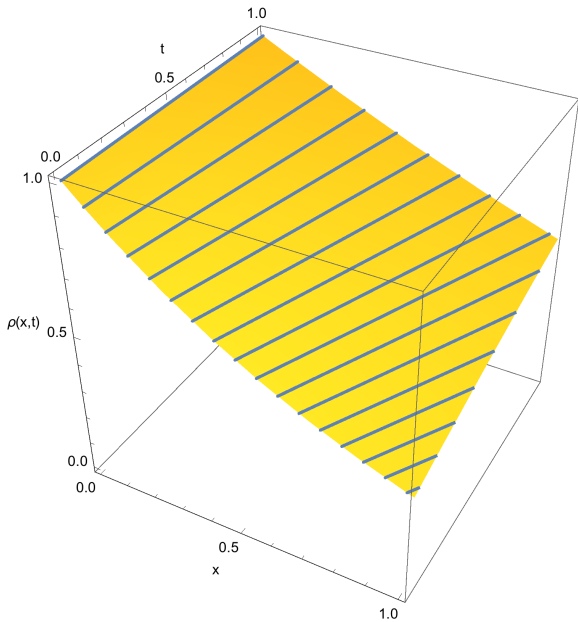
## Case: $f'(s) < 0$

If  $f(s)$  is decreasing then characteristics do not intersect for  $t \geq 0$ .



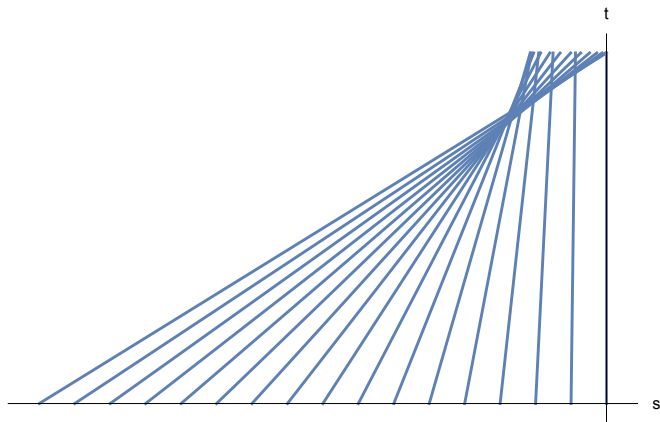
The density of vehicles is well-defined for  $t \geq 0$ .

Case:  $f'(s) < 0$



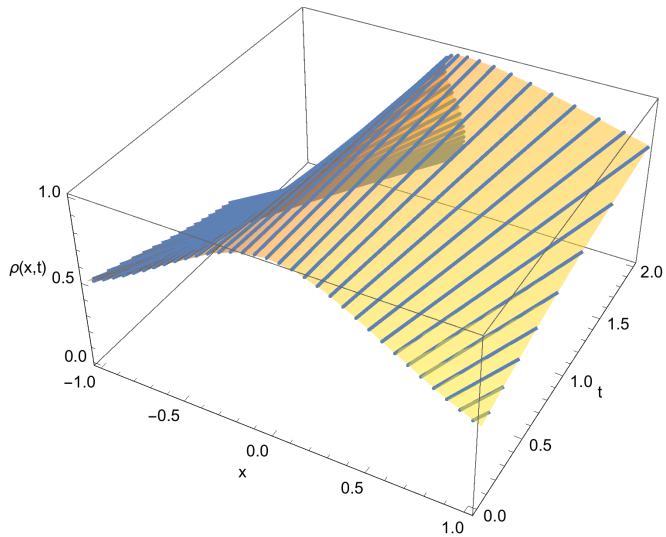
## Case: $f'(s) > 0$

If  $f(s)$  is increasing then characteristics intersect for some  $t \geq 0$ .

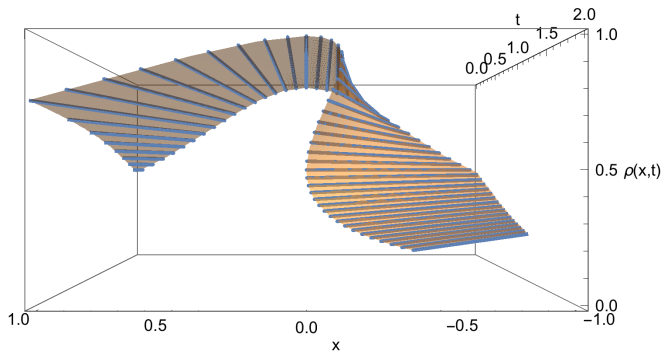


This implies the density takes on two different values. This phenomenon is known as a **shock**.

Case:  $f'(s) > 0$



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**Question:** when does the shock occur?

## Time of Shock (1 of 2)

- ▶ The shock occurs at the earliest time for which there is a vertical tangent line to the integral surface.

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- ▶ Suppose the shock occurs at  $t = t_0$ , then

$$x(s) = u_{\max} \left( 1 - \frac{2f(s)}{\rho_{\max}} \right) t_0 + s$$
$$\rho(s) = f(s).$$

We must find the tangent vector to this parametrically defined curve.

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- ▶ Using the chain rule for derivatives,

$$\frac{d\rho}{dx} = \frac{d\rho/ds}{dx/ds} = \frac{f'(s)}{1 - \frac{2u_{\max}t_0}{\rho_{\max}} f'(s)}.$$

## Time of Shock (2 of 2)

$$\frac{d\rho}{dx} = \frac{f'(s)}{1 - \frac{2u_{\max}t_0}{\rho_{\max}} f'(s)}$$

Thus the shock occurs when  $f'(s) \neq 0$  and

$$1 - \frac{2u_{\max}t_0}{\rho_{\max}} f'(s) = 0 \iff t_0 = \frac{\rho_{\max}}{2u_{\max} f'(s)}.$$

## Example

Suppose  $\rho(x, 0) = f(x) = \frac{\rho_{\max}/3}{1+x^2}$ .

1. Where is the point of maximum density initially?
2. Where is the point of maximum density at time  $t > 0$ ?
3. When does the first shock occur?
4. Where does the first shock occur?

## Solution (1 of 2)

$$f(s) = \frac{\rho_{\max}/3}{1 + s^2}$$
$$f'(s) = -\frac{2\rho_{\max}}{3} \frac{x}{(1 + s^2)^2}$$

At  $t = 0$  the maximum density is at  $x_0 = 0$ .

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**Recall:** for each constant  $x_0$  the solution has constant value  $f(x_0)$  along the line

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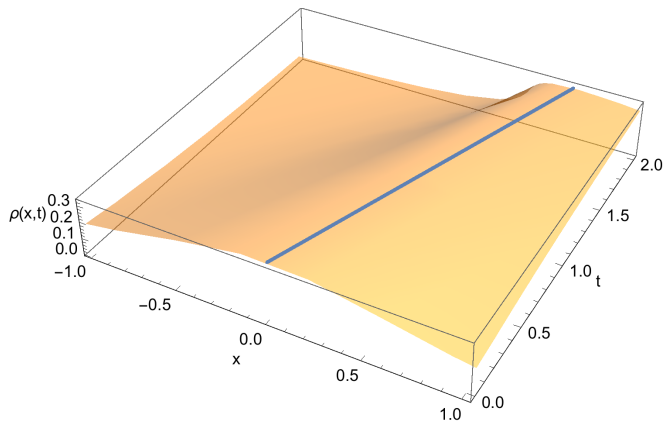
$$x = u_{\max} \left( 1 - \frac{2f(x_0)}{\rho_{\max}} \right) t + x_0.$$

Thus if  $x_0 = 0$  the maximum density at  $t > 0$  is at

$$x = u_{\max} \left( 1 - \frac{2f(0)}{\rho_{\max}} \right) t = \frac{u_{\max} t}{3}.$$

The point of maximum density is moving to the right at  $1/3$  the maximum speed of the vehicles.

# Tracking the Maximum Density



## Solution (2 of 2)

The first shock arises when

$$t_0 = \frac{\rho_{\max}}{2u_{\max}f'(s)}.$$

The maximum of  $f'(s)$  occurs at  $s = -1/\sqrt{3}$  and thus

$$t_0 = \frac{\rho_{\max}}{2u_{\max}f'(-1/\sqrt{3})} = \frac{4}{u_{\max}\sqrt{3}},$$

is the time of the first shock.

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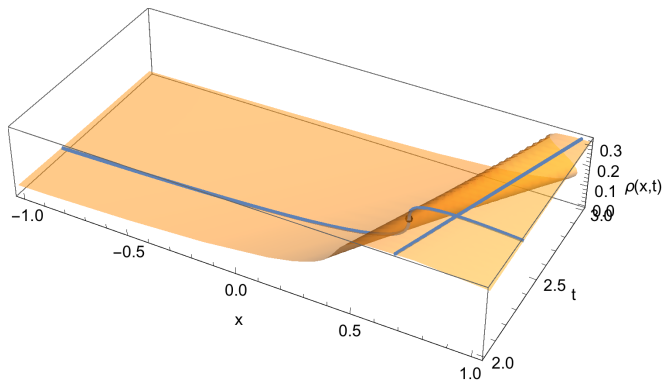
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is the time of the first shock.

The location of the first shock is

$$\begin{aligned}x &= u_{\max} \left( 1 - \frac{2f(s)}{\rho_{\max}} \right) t + s \\ &= u_{\max} \left( 1 - \frac{2f(-1/\sqrt{3})}{\rho_{\max}} \right) \frac{4}{u_{\max}\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.\end{aligned}$$

# Illustration



**Note:** the shock occurs at a level below the maximum density.

# Homework

- ▶ Read Section 2.3
- ▶ Exercises: 15–20