

Preliminary Definitions and Examples

MATH 467 *Partial Differential Equations*

J Robert Buchanan

Department of Mathematics

Fall 2022

What is a PDE?

Definition

A **partial differential equation** (PDE) is an equation involving an unknown function u , of two or more independent variables t , x , y , *etc* and the partial derivatives of u .

Examples

$$u_t + u u_y = 0$$

$$u_{xx} + u_{yy} = f(x, y)$$

$$u_{xx} + x y^3 u_{yy} - e^x u_z + x y u = 0$$

$$u_t - u_{xx} + (u_x)^2 = 0$$

$$u_t + c u u_x + u_{xxx} = 0$$

Partial Derivative Notation

$$u_t = \frac{\partial u}{\partial t}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x}$$

⋮

Description of PDEs

PDEs can be described as being

- ▶ linear or nonlinear (or in some cases semilinear or quasilinear),
- ▶ homogeneous or nonhomogeneous,
- ▶ first order, second order, third order, *etc.*

Order

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Find the order of each of the following PDEs.

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Linearity

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A PDE is **linear** if all the terms of the PDE are linear in the unknown and its partial derivatives. Otherwise the PDE is **nonlinear**.

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Determine which of the following PDEs are linear or nonlinear.

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$$u_{xx} + u_{yy} = f(x, y)$$

$$u_{xx} + x y^3 u_{yy} - e^x u_z + x y u = 0$$

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Solution to a PDE

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Consider the second-order, linear PDE

$$u_{xy} = 0 \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Verify that $u(x, y) = F(x) + G(y)$ is a solution to the PDE where F and G are arbitrary differentiable functions defined for all real numbers.

Common Generic PDEs

We will primarily focus on properties and solutions of the following two types of PDE.

First-order Linear:

$$a(x, t)u_x + b(x, t)u_t + c(x, t)u = g(x, t)$$

Second-order Linear:

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

where $A, B, C, D, E, F,$ and G are functions of (x, t) .

Note: a linear PDE is called **homogeneous** if $g(x, t) \equiv 0$ or $G(x, t) \equiv 0$, otherwise it is **nonhomogeneous**.

Principle of Superposition

Theorem

If u_i is a solution of the second-order, linear partial differential equation

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G_i$$

for $i = 1, 2, \dots, n$ on a domain $\Omega \subset \mathbb{R}^2$, then for any constants c_1, c_2, \dots, c_n the function $u = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$ is a solution of

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = \sum_{i=1}^n c_i G_i$$

on Ω .

Principle of Subtraction

Theorem

If $u_1(x, t)$ and $u_2(x, t)$ are solutions to

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

then $u(x, t) \equiv u_1(x, t) - u_2(x, t)$ is a solution to

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = 0.$$

Remark: any two solutions to a nonhomogeneous PDE differ by a solution to the associated homogeneous PDE.

Initial and Boundary Conditions

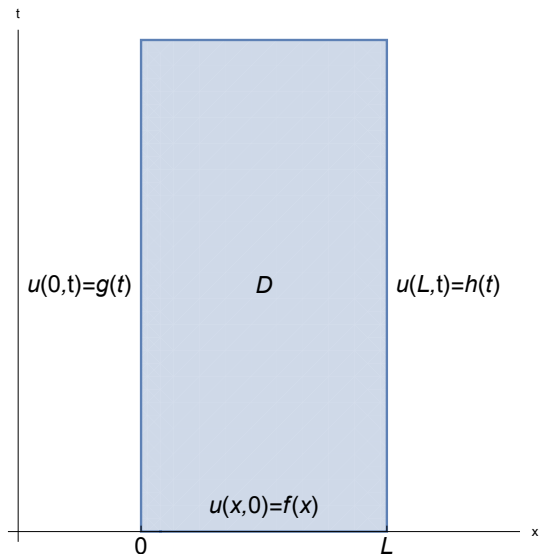
- ▶ If $u(x, t)$ is the solution to a PDE which satisfies the requirement that $u(x, 0) = f(x)$, then this requirement is called an **initial condition**.
- ▶ If the domain of the solution $u(x, t)$ is $D = \{(x, t) : t \geq 0, 0 \leq x \leq L\}$ and

$$u(0, t) = g(t)$$

$$u(L, t) = h(t),$$

then these requirements are called **boundary conditions**.

Illustration



Initial Boundary Value Problems

Definition

A PDE along with specified initial and boundary conditions is referred to as an **initial boundary value problem** (IBVP). If only boundary conditions are specified (perhaps because u is independent of t) the PDE is called a **boundary value problem** (BVP).

Definition

An IBVP is **well-posed** if it has a unique solution and that solution depends continuously on the initial and boundary conditions. Otherwise the PDE is **ill-posed**.

Example

Consider the PDE:

$$u_{xx} + u_{yy} = 0.$$

Verify that the following functions solve the PDE.

$$u_1(x, y) = x + y$$

$$u_2(x, y) = x^2 - y^2$$

$$u_3(x, y) = e^x \cos y$$

$$u_4(x, y) = \ln(x^2 + y^2)$$

$$u_5(x, y) = \frac{x}{x^2 + y^2}$$

Example: BVP

Consider the BVP:

$$u_{xx} + u_{yy} = 0$$

$$u(x, y) = 0 \quad \text{for } x^2 + y^2 = 1.$$

Find its solution.

Example: IBVP

Consider the IBVP:

$$u_t - u_{xx} = 0 \quad \text{for } t \geq 0, 0 \leq x \leq \pi$$

$$u(x, 0) = 100 + \sin x \quad (\text{IC})$$

$$u(0, t) = 100 \quad (\text{BC for } x = 0)$$

$$u(\pi, t) = 100 \quad (\text{BC for } x = \pi)$$

1. Show that $u_1(x, t) = t + x^2/2$ solves the PDE.

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1. Show that $u_1(x, t) = t + x^2/2$ solves the PDE.
2. Show that $u_2(x, t) = A + Be^{-t} \sin x$ solves the PDE.
3. Choose A and B appropriately so that $u_2(x, t)$ solves the IBVP.

Classification of PDEs

Recall the general second-order PDE:

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

where A, B, C, D, E, F , and G are functions of $(x, t) \in D$. The PDE is said to be

elliptic on D if $4AC - B^2 > 0$ for all $(x, t) \in D$,

parabolic on D if $4AC - B^2 = 0$ for all $(x, t) \in D$,

hyperbolic on D if $4AC - B^2 < 0$ for all $(x, t) \in D$.

Examples

Classify each of the following PDEs as either **elliptic**, **parabolic**, or **hyperbolic**.

$$u_{tt} + u_{xx} + u_t - 2u_x = f(x, t)$$

$$u_{tt} - u_{tx} + 2u_{xx} + e^x u_t - \sin(tx)u_x = f(x, t)$$

$$u_{tt} - 2u_{tx} + u_{xx} + 2u_t - u_x = 0$$

$$u_{tt} + t u_{xx} = 0$$

$$u_{xx} - \beta u_t + \alpha u_x + \rho u + f(x, t) = u_t$$

Homework

- ▶ Read Sections 1.1 and 1.2
- ▶ Exercises: 1–6