Quadratic Forms
Math 422

**Definition 1** A **quadratic form** is a function $f : \mathbb{R}^n \to \mathbb{R}$ of form

$$f(x) = x^T Ax,$$

where $A$ is an $n \times n$ symmetric matrix.

**Example 2** $f(x, y) = 2x^2 + 3xy - 4y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$

Note that the Euclidean inner product (dot product) of two (column) vectors $a$ and $b$ can be expressed in terms of matrix multiplication as $\langle a, b \rangle = b^T a$.

Thus, a quadratic form can be expressed in terms of the Euclidean inner product as

$$x^T Ax = \langle Ax, x \rangle = \langle x, Ax \rangle.$$

Let $S^{n-1}$ denote the unit $(n-1)$-dimensional sphere in $\mathbb{R}^n$, i.e., relative to the Euclidean inner product

$$S^{n-1} = \{ x \in \mathbb{R}^n : \langle x, x \rangle = 1 \}.$$

Since $S^{n-1}$ is a closed and bounded subset of $\mathbb{R}^n$, continuous functions on $S^{n-1}$ attain their maximum and minimum values.

**Question #1:** For $x \in S^{n-1}$, what are the maximum and minimum values of a quadratic form $x^T Ax$?

**Theorem 3** Let $A$ be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

1. $\lambda_1 \geq x^T Ax \geq \lambda_n$ for all $x \in S^{n-1}$.
2. If $x_1 \in S^{n-1}$ is an eigenvalue associated with $\lambda_1$, then $\lambda_1 = x_1^T Ax_1$.
3. If $x_n \in S^{n-1}$ is an eigenvalue associated with $\lambda_n$, then $\lambda_n = x_n^T Ax_n$.

The maximum and minimum of a quadratic form $x^T Ax$ can be found by computing the largest and smallest eigenvalue of $A$. The maximum (respectively, minimum) will always be attained at diametrically opposite points on the unit sphere $\pm \frac{x}{\|x\|}$, where $x$ is any eigenvector associated with $\lambda_1$ (respectively, $\lambda_n$).

**Example 4** Consider $f(x_1, x_2) = 2x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Since the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = -1$, the maximum and minimum values of $f$ on the unit circle $S^1$ are 1 and $-1$, respectively. Furthermore, the maximum value is attained at the eigenvectors on $S^1$ associated with $\lambda_1 = 1$, namely $\pm \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$; the minimum value is attained at the eigenvectors on $S^1$ associated with $\lambda_2 = -1$, namely $\pm \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

**Question #2:** Under what conditions is the quadratic form $x^T Ax > 0$ for all $x \neq 0$?
Definition 5 \( x^T A x \) is positive definite iff \( x^T A x > 0 \) for all \( x \neq 0 \) and \( x^T A x = 0 \) iff \( x = 0 \). A symmetric matrix is positive definite iff \( x^T A x \) is positive definite.

Example 6 The Euclidean inner product is a positive definite quadratic form since
\[
x_1^2 + \cdots + x_n^2 = (x, x) = x^T x = x^T I x.
\]

Theorem 7 A symmetric matrix \( A \) is positive definite iff all eigenvalues of \( A \) are positive.

Example 8 The matrix \( A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \) is positive definite since the eigenvalues of \( A \) are \( \lambda_1 = 8 \), \( \lambda_2 = 2 \) and \( \lambda_3 = 2 \). Note that if \( x \neq 0 \), then \( x^T A x = 2x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 > 0 \).

Definition 9 For \( 1 \leq k \leq n \), the \( k \)th principal submatrix of an \( n \times n \) matrix \( A = [a_{ij}] \) is
\[
\begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}.
\]

Theorem 10 A symmetric matrix \( A \) is positive definite iff every principal subdeterminant of \( A \) is positive.

Example 11 The principal subdeterminants of the matrix \( A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix} \) are \( \det [2] = 2 \), \( \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3 \) and \( \det A = 1 \). Since all are positive, the quadratic form \( x^T A x \) is positive definite.

Question #3: If \( x^T A x \) is a quadratic form with non-diagonal \( A \), under what conditions does there exist an orthogonal change variables \( x = Py \) so that \( (Py)^T A (Py) = y^T (P^T A P) y \) has no cross-terms?

Definition 12 An \( n \times n \) matrix \( A \) is orthogonally diagonalizable iff there exists an orthogonal matrix \( P \) such that \( P^T A P \) is a diagonal matrix.

Theorem 13 If \( A \) is an \( n \times n \) matrix, then the following are equivalent:

1. \( A \) is orthogonally diagonalizable.
2. \( A \) has an orthonormal set of \( n \) eigenvectors.
3. \( A \) is symmetric.

Theorem 14 If \( A \) is symmetric, then

1. The eigenvalues of \( A \) are real numbers.
2. Eigenvectors from different eigenspaces are orthogonal with respect to the Euclidean inner product.

Use the following procedure to orthogonally diagonalize \( A \):

Example 15 1. Find a basis for each eigenspace of \( A \).
2. Apply Gram-Schmidt and obtain an orthonormal basis for each eigenspace.
3. Form the matrix \( P \) whose columns are the basis vectors constructed in step 2.
Example 16 Consider the matrix \( A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \) whose eigenvalues are \( \lambda_1 = 8 \), \( \lambda_2 = 2 \) and \( \lambda_3 = 2 \).

Canonical bases for the eigenspaces are \( \{ x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \} \)

\( \leftrightarrow 8 \) and \( \{ x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \} \leftrightarrow 2. \) Note that \( \langle x_1, x_2 \rangle = \langle x_1, x_3 \rangle = 0. \) Applying Gram-Schmidt gives

\[ \{ v_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \} \] and

\[ \{ v_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, v_3 = \frac{x_3 - \langle x_3, v_2 \rangle v_2}{\|x_3 - \langle x_3, v_2 \rangle v_2\|} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \}. \] The matrix \( P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \)

and \( P^T A P = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \) is a diagonal matrix.

Theorem 17 Let \( x^T A x \) be a quadratic form in variables \( x_1, \ldots, x_n \). Let \( P \) be an orthogonal matrix that orthogonally diagonalizes \( A \). If \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \) and \( y_1, \ldots, y_n \) are new variables such that \( x = Py \), then

\[ x^T A x = y^T (P^T A P) y = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \]

and

\[ P^T A P = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \]

Example 18 Let \( A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \). By the calculations in Example 16, \( 2x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 = x^T A x = y^T (P^T A P) y = 8y_1^2 + 2y_2^2 + 2y_3^2 \).

Question #4: If \( x^T A x \) is a quadratic form in two or three variables and \( c \) is a constant, what does the graph of the level set \( x^T A x = c \) look like?

Theorem 19 If \( x^T A x \) is a quadratic form in two variables and \( c \) is a constant, the level curve given by \( x^T A x = c \) is a conic. If \( x^T A x \) is a quadratic form in three variables and \( c \) is a constant, the level surface given by \( x^T A x = c \) is a quadric.

Example 20 In Example 4, let \( P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \); then

\[ P^T A P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]. The level curve given by \( 2x_1x_2 = 1 \) is the hyperbola \( y_1^2 - y_2^2 = 1 \) since

\[ 2x_1x_2 = x^T A x = y^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} y = y_1^2 - y_2^2. \]

From Example 18 we observe that the level surface \( 2x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 = 1 \) is the ellipsoid \( 8y_1^2 + 2y_2^2 + 2y_3^2 = 1 \).
Exercise 21  Since the quadratic form in Example 11 is positive definite, the quadric given by \( x^T A x = 1 \) is an ellipsoid. Eliminate the cross-terms by performing an orthogonal change of variables. Express this ellipsoid in the standard form \( \frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} + \frac{y_3^2}{c^2} = 1 \).

Definition 22  A quadratic form \( x^T A x \) is non-degenerate if all eigenvalues of \( A \) are non-zero.

Definition 23  The signature of a non-degenerate quadratic form \( x^T A x \), denoted by \( \text{sig}(A) \), is the number of negative eigenvalues of \( A \).

Theorem 24  Let \( x^T A x \) be a non-degenerate quadratic form in two variables.

1. If \( \text{sig}(A) = 0 \), then \( x^T A x = 1 \) is an ellipse.
2. If \( \text{sig}(A) = 1 \), then \( x^T A x = 1 \) is a hyperbola.

Theorem 25  Let \( x^T A x \) be a non-degenerate quadratic form in three variables.

1. If \( \text{sig}(A) = 0 \), then \( x^T A x = 1 \) is an ellipsoid.
2. If \( \text{sig}(A) = 1 \), then \( x^T A x = 1 \) is a hyperboloid of one sheet.
3. If \( \text{sig}(A) = 2 \), then \( x^T A x = 1 \) is a hyperboloid of two sheets.