Minkowski Space

For simplicity, we consider 2-dimensional space-time, or *Minkowski space*, which is the *pseudo inner product space*

\[ \mathbb{R}^2_1 = \{(t, x) : t, x \in \mathbb{R}\} \]

with *pseudo inner product* defined by

\[ \langle (t_1, x_1), (t_2, x_2) \rangle = t_1t_2 - x_1x_2. \]

The *Minkowski norm*

\[ \| (t, x) \| = \sqrt{t^2 - x^2} \]

ranges over all non-negative real and positive imaginary values.

Curves of constant Minkowski norm \( a \) satisfy the equation

\[ t^2 - x^2 = a^2. \] (1)

The parameter \( a \) determines three families of such curves:

- When \( a = 0 \), (1) defines the *light cone* \( x = \pm t \).
- When \( a \in \mathbb{R}^+ \), (1) defines a *real hyperbolic circle of radius* \( a \), which is the hyperbola \( t^2 - x^2 = a^2 \) inside the light cone.
- When \( a = ib \in i\mathbb{R}^+ \), (1) defines an *imaginary hyperbolic circle of radius* \( ib \), which is the hyperbola \( x^2 - t^2 = b^2 \) outside the light cone.

- *Isotropic vectors* have zero Minkowski norm and live on the light-cone.
- *Time-like vectors* have positive real Minkowski norm and live inside the light-cone.
- *Space-like vectors* have positive imaginary Minkowski norm and live outside the light-cone.

Let \( C: (t(u), x(u)), a \leq u \leq b \), be a parametrized curve in \( \mathbb{R}^2_1 \) and define the *hyperbolic arc length* \( s \) of \( C \) to be

\[ s = \int_a^b \| (t'(u), x'(u)) \| du = \int_a^b \sqrt{(t')^2 - (x')^2} du. \]

**Hyperbolic Circles**

Recall that Euclidean angle \( \theta \) measures the arc length along the unit circle in \( \mathbb{R}^2 \) from \((1,0)\) to \((\cos \theta, \sin \theta)\). Note that the area of the sector subtending the arc from \((\cos \theta, -\sin \theta)\) to \((\cos \theta, \sin \theta)\) is also \( \theta \). On the other hand, the arc length from \((1,0)\) to \((\cosh \theta, \sinh \theta)\) along the real hyperbolic circle \( t^2 - x^2 = 1 \) is \( \theta i \), but we’d like it to be \( \theta \). Thankfully, the area of the real hyperbolic sector subtending the hyperbolic arc from \((\cosh \theta, -\sinh \theta)\) to \((\cosh \theta, \sinh \theta)\) is exactly \( \theta \). So let’s redefine Euclidean angle \( \theta \) to be the area of the sector subtending the arc from \((\cos \theta, -\sin \theta)\) to \((\cos \theta, \sin \theta)\); then analogously, define the *hyperbolic angle* \( \theta \) to be the area of the real hyperbolic sector subtending the real hyperbolic arc from \((\cosh \theta, -\sinh \theta)\) to \((\cosh \theta, \sinh \theta)\). Note that \((\cos \theta, \sin \theta)\) and \((-\cos \theta, -\sin \theta)\) are antipodes on the unit circle, and likewise \((\cosh \theta, \sinh \theta)\) and \((-\cosh \theta, -\sinh \theta)\) are antipodes on the real hyperbolic circle \( t^2 - x^2 = 1 \).

Parametrize the imaginary hyperbolic circle \( C: x^2 - t^2 = b^2 \) by \( t = b \sinh \theta, x = b \cosh \theta \). Then the arc length function along \( C \) is

\[ s(\theta) = \int_0^\theta \| (b \cosh u, b \sinh u) \| du = \sqrt{b^2} \int_0^\theta du = b \theta \]
so that $\theta = s/b$. Now substituting for $\theta$ in terms of $s$ reparametrizes $C$ by arc length and gives the position function

$$r(s) = b \left( \sinh \left( \frac{s}{b} \right), \cosh \left( \frac{s}{b} \right) \right)$$

with velocity

$$v(s) = \left( \cosh \left( \frac{s}{b} \right), \sinh \left( \frac{s}{b} \right) \right)$$

and constant speed

$$\|v(s)\| = \sqrt{\cosh^2 \left( \frac{s}{b} \right) - \sinh^2 \left( \frac{s}{b} \right)} = 1.$$

Thus the unit tangent vector field $T(s)$ along $C$ is simply the velocity vector field $v(s)$ along $C$, and the curvature $\kappa$ of $C$ is the magnitude of $T'(s)$, i.e., the instantaneous rate at which $T$ changes direction. Thus

$$T'(s) = v'(s) = \frac{1}{b} \left( \sinh \left( \frac{s}{b} \right), \cosh \left( \frac{s}{b} \right) \right)$$

and the curvature

$$\kappa = \|T'(s)\| = \frac{1}{b} \sqrt{\sinh^2 \left( \frac{s}{b} \right) - \cosh^2 \left( \frac{s}{b} \right)} = i = -\frac{1}{bi}$$

is the negative reciprocal of the imaginary radius. Note that unlike Euclidean circular motion, in which the acceleration and position have opposite directions, the acceleration and position of imaginary hyperbolic circular motion have the same direction:

$$b(s) = v'(s) = \frac{1}{b^2} r(s).$$

Nevertheless, hyperbolic and Euclidean circles have similar properties.

**Exercises**

1. Show that the length of the arc along the real hyperbolic unit circle from $(1,0)$ to $(\cosh \theta, \sinh \theta)$ is $\theta i$.

2. Show that the area of the real hyperbolic unit sector with angle $2\theta$, i.e., the plane region bounded by the lines $x = \pm t \tanh \theta$ and $t^2 - x^2 = 1$, is exactly $\theta$. (Your integration will require a hyperbolic trigonometric substitution.)

3. Given $a > 0$, parametrize the real hyperbolic circle $C: t^2 - x^2 = a^2$ by $t = a \cosh \theta$, $x = a \sinh \theta$. Reparametrize by arc length and show that the curvature $\kappa = \frac{1}{a}$ and that acceleration and position have opposite directions.

**The Poincaré Group**

Recall that an isometry (fixing the origin) is a norm preserving linear transformation. Plane Euclidean isometries are rotations $\rho_\theta$ about the origin through angle $\theta$ and reflections $\sigma_\theta$ in lines through the origin with inclination $\theta$. Euclidean rotations fix circles centered at the origin and send lines through the origin to lines through the origin; reflections fix their reflecting lines point-wise. Rotations $\rho_\theta$ are represented by orthogonal matrices with determinant $+1$:

$$[\rho_\theta] = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$  

Reflections $\sigma_\theta$ are represented by orthogonal matrices with determinant $-1$:

$$[\sigma_\theta] = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{bmatrix};$$

and in particular,

$$[\sigma_0] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
Euclidean isometries are represented by elements of the orthognal group $O(2)$, which is the union of two disjoint components

$$[\rho_0] = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{and} \quad [\rho_0][\sigma_0] = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}.$$ 

The component

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \cos \theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

parametrizes the circle

$$C_1 : u_1^2 + u_2^2 = 2$$
in the 2-plane spanned by

$$B_1 = \left\{ u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$ 

Note that the trivial rotation

$$[\rho_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sqrt{2} u_1 + 0 u_2$$
is the point $(\sqrt{2}, 0)$ on this circle in the basis $B_1$. Similarly, the component

$$\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} = \cos \theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
parametrize the circle

$$C_2 : u_3^2 + u_4^2 = 2$$
in the 2-plane spanned by

$$B_2 = \left\{ u_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad u_4 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$ 

Since $B_1 \cup B_2$ is linearly independent in $\mathbb{R}^{2\times 2}$, $C_1 \cap C_2 = \varnothing$ and $C_1 \cup C_2 = O(2)$.

The situation in Minkowski space is similar but a bit more complicated. Here the isometries are hyperbolic rotations and hyperbolic reflections. The group of all such transformations, called the Poincaré group $O(1, 1)$, has four connected components, which appear as the branches of two hyperbolas in $\mathbb{R}^{2\times 2}$. A hyperbolic rotation $R_\theta$ through angle $\theta$ is represented by the matrix

$$[R_\theta] = \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix}$$
and is given in coordinates by

$$R_\theta (t, x) = (t \cosh \theta + x \sinh \theta, t \sinh \theta + x \cosh \theta).$$

Note that $R_\theta$ fixes hyperbolic circles: If $(\tilde{t}, \tilde{x}) = R_\theta (t, x)$, then

$$(\tilde{t})^2 - (\tilde{x})^2 = (t \cosh \theta + x \sinh \theta)^2 - (t \sinh \theta + x \cosh \theta)^2 = t^2 - x^2.$$ 

Furthermore, $R_\theta$ sends lines through the origin to lines through the origin since

$$R_\theta (a, 0) = a (\cosh \theta, \sinh \theta).$$

Denote reflection in the $t$-axis by

$$S_0 (t, x) = (t, -x)$$
and reflection in the $x$-axis by

$$S_\infty (t, x) = (-t, x).$$
Then $S_0$ and $S_{\infty}$ fix hyperbolic circles since
\[ t^2 - x^2 = t^2 - (-x)^2 = (-t)^2 - x^2. \]

Let $l_0$ be a line through the origin with inclination $\theta \neq \pm \pi/4$, and let $R_{l_0}$ be the hyperbolic rotation that rotates $l_0$ onto the $t$-axis if $-\pi/4 < \theta < \pi/4$, and rotates $l_0$ onto the $x$-axis if $\pi/4 < \theta < 3\pi/4$. The 
*hyperbolic reflection in line* $l$ is the composition
\[ S_m = \begin{cases} R_{l_0}^{-1} S_0 R_{l_0}, & \text{if } -\pi/4 < \theta < \pi/4 \\ R_{l_0}^{-1} S_{\infty} R_{l_0}, & \text{if } \pi/4 < \theta < 3\pi/4. \end{cases} \]

Hyperbolic reflections fix hyperbolic circles and fix their reflecting lines point-wise. Somewhat surprisingly perhaps, there are no hyperbolic reflections in the lines $x = \pm t$ (see Exercise 7 below). The reflections $S_0$ and $S_{\infty}$ are represented by the matrices
\[ [S_0] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad [S_{\infty}] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \]

Minkowski isometries are represented by elements of the Poincaré group $O(1,1)$, which is the union of four mutually disjoint components:
\[ [R_{\theta}] = \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix}, \quad [R_0] [S_0] [S_{\infty}] = \begin{bmatrix} -\cosh \theta & -\sinh \theta \\ -\sinh \theta & -\cosh \theta \end{bmatrix}. \]

The components
\[ \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix} = \cosh \theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sinh \theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

and
\[ \begin{bmatrix} -\cosh \theta & -\sinh \theta \\ -\sinh \theta & -\cosh \theta \end{bmatrix} = -\cosh \theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \sinh \theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

form the two branches of the hyperbola
\[ H_1 : u_1^2 - u_4^2 = 2 \]

in the 2-plane spanned by
\[ \mathcal{B}_1' = \left\{ u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \ u_4 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}. \]

The trivial hyperbolic rotation
\[ [R_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sqrt{2} u_1 + 0 u_4 \]

is the point $(\sqrt{2},0)$ on this hyperbola in the basis $\mathcal{B}_1$. The components
\[ \begin{bmatrix} \cosh \theta & -\sinh \theta \\ \sinh \theta & -\cosh \theta \end{bmatrix} = \cosh \theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \sinh \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

and
\[ \begin{bmatrix} -\cosh \theta & \sinh \theta \\ -\sinh \theta & \cosh \theta \end{bmatrix} = -\cosh \theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \sinh \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

form the two branches of the hyperbola
\[ H_2 : u_3^2 - u_2^2 = 2 \]

in the 2-plane spanned by
\[ \mathcal{B}_2' = \left\{ u_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \ u_2 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}. \]

Since $\mathcal{B}_1' \cup \mathcal{B}_2'$ is linearly independent in $\mathbb{R}^{2\times2}$, $H_1 \cap H_2 = \emptyset$ and $H_1 \cup H_2 = O(1,1)$.
Exercises

4. Find the matrices \( R \), \( R S_0 \), and \( R S_\infty \).

5. Prove that a hyperbolic reflection fixes its reflecting line point-wise.

6. Prove that \( S_0 \) and \( S_\infty \) are the only hyperbolic reflections that are also Euclidean reflections.

7. Prove that a Minkowski norm preserving linear transformation that fixes the line \( x = t \) point-wise is the identity transformation. Prove the analogous statement for the line \( x = -t \).

Special Relativity

The speed of light \( c \approx 3 \times 10^8 \) m/sec.

- An event is a point \((t, x)\) in space-time.
- The world-line of a particle \(P\) is a parametrized curve \(r(t) = (ct, x(t))\).
- The relative velocity of \(P\) along its world line is \(v(t) = (c, x'(t))\).
- The ordinary velocity of \(P\) is \(x'(t)\).
- The relative speed of \(P\) along its world line is \(||v|| = \sqrt{c^2 - (x')^2}\).
- The ordinary speed of \(P\) is \(|x'|\).

Physical Assumption 1: The ordinary speed of a particle cannot exceed the speed of light.

Hence \((x')^2 \leq c^2\) and

\[ ||v||^2 = c^2 - (x')^2 \geq 0. \]

Thus vectors tangent to the world-line of a particle in motion are either time-like or isotropic.

Physical Assumption 2: A particle traveling at the speed of light has zero mass.

- The world-line of a particle at rest is a horizontal line inside the light cone.
- The world-line of a particle with non-zero mass is a curve inside the light cone.
- The world-line of a particle with non-zero mass and constant speed is a line inside the light cone.
- The world-line of a photon is a line on the light cone.

Consider the world-line \(r(t) = (ct, x(t))\) of a particle \(P\) with non-zero mass and constant ordinary velocity \(v\) in the positive \(x\)-direction. The relative velocity of \(P\) is \((c, v)\) and its relative speed is \(\sqrt{c^2 - v^2}\). The arc length function \(s\) for the world-line of \(P\) is

\[ s(t) = \int_0^t ||(c, v)|| \, du = \sqrt{c^2 - v^2} \int_0^t \, du = t\sqrt{c^2 - v^2}; \]

hence

\[ t = \frac{s}{\sqrt{c^2 - v^2}} = \frac{s/c}{\sqrt{1 - v^2/c^2}}. \]

The proper elapsed time of \(P\) is the quantity

\[ \frac{s}{c} = t\sqrt{1 - v^2/c^2}. \]
If $P$ is at rest, for example, its proper elapsed time is $t$.

**Lorentz Transformations**

“Lorentz transformations” are special elements of the Poincaré group that change coordinates from reference frame $\bar{K} (\vec{c}t, \vec{x})$ to reference frame $K (ct, x)$ or vice versa as $\bar{K}$ moves along a straight line with constant velocity relative to $K$.

**Physical Assumption 3:** *The speed of light $c$ is the same in every frame of reference.*

Assume that $\bar{K}$ moves in the positive $x$ direction in $K$ with constant speed $v$. A *Lorentz transformation* is a hyperbolic change of coordinates

$$
\begin{bmatrix}
ct \\
x
\end{bmatrix} =
\begin{bmatrix}
\alpha & \beta \\ 
\gamma & \delta
\end{bmatrix}
\begin{bmatrix}
\vec{c}t \\
\vec{x}
\end{bmatrix}
.$$  

(3)

Note that

$$A =
\begin{bmatrix}
\alpha & \beta \\ 
\gamma & \delta
\end{bmatrix}
$$

lies in the component of $I \in O (1, 1)$ since $A \rightarrow I$ as $v \rightarrow 0$. Hence $A$ is a hyperbolic rotation

$$R_\theta =
\begin{bmatrix}
\cosh \theta & \sinh \theta \\ 
\sinh \theta & \cosh \theta
\end{bmatrix}
$$

and equation (3) becomes

$$
\begin{bmatrix}
ct \\
x
\end{bmatrix} =
\begin{bmatrix}
\cosh \theta & \sinh \theta \\ 
\sinh \theta & \cosh \theta
\end{bmatrix}
\begin{bmatrix}
\vec{c}t \\
\vec{x}
\end{bmatrix}
.$$  

(4)

**Example.** Let $\theta = \ln 2$; then $\cosh \theta = \frac{5}{4}$ and $\sinh \theta = \frac{3}{4}$. Thus

$$
\begin{bmatrix}
5 \\
3
\end{bmatrix} =
\begin{bmatrix}
5/4 & 3/4 \\
3/4 & 5/4
\end{bmatrix}
\begin{bmatrix}
4 \\
0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 \\
4
\end{bmatrix} =
\begin{bmatrix}
5/4 & 3/4 \\
3/4 & 5/4
\end{bmatrix}
\begin{bmatrix}
-3 \\
5
\end{bmatrix}.
$$

This particular hyperbolic rotation moves the point $(4, 0)$ “counterclockwise” along the hyperbola $t^2 - x^2 = 16$ to the point $(5, 3)$, and the point $(-3, 5)$ “clockwise” along the hyperbola $x^2 - t^2 = 16$ to the point $(0, 4)$. The flows along these two hyperbolas asymptotically approach the light cone $x = t$ in the second quadrant (see Figure 1).

![Hyperbolic rotations fix hyperbolas.](image)

Figure 1. Hyperbolic rotations fix hyperbolas.

**Lorentz transformations change coordinates of**

- *time-like vectors (inside the light cone)* from $\bar{K}$-coordinates to $K$-coordinates and
- *space-like vectors (outside the light cone)* from $K$-coordinates to $\bar{K}$-coordinates.

Let’s investigate the motion of a particle $P$ with non-zero mass positioned at the origin $\bar{O}$ in the moving frame $\bar{K}$ as it moves in the positive $x$-direction in frame $K$ with constant speed $v$. Since $P$ is at rest in frame $\bar{K}$, its world line in $\bar{K}$ is the parametrized curve $(ct, 0)$ contained in the $t$ axis. But when viewed from frame
its world line has positive slope inside the light cone and is parameterized by \((ct, x) = (c\cosh \theta, c\sinh \theta)\) via equation (4). Now dividing second components by first components gives

\[
\tanh \theta = \frac{x}{c t} = \frac{v t}{c} = \frac{v}{c}.
\]

(5)

Now using the fact that \(\cosh \theta > 0\), solve for \(\cosh \theta\) in the identity

\[
1 = \cosh^2 \theta - \sinh^2 \theta = \cosh^2 \theta (1 - \tanh^2 \theta)
\]

and obtain

\[
\cosh \theta = \frac{1}{\sqrt{1 - \tanh^2 \theta}} = \frac{1}{\sqrt{1 - v^2/c^2}}.
\]

Combining this with equation (5) gives

\[
\sinh \theta = \frac{v/c}{\sqrt{1 - v^2/c^2}},
\]

and substituting in (4) we obtain

\[
t = \frac{1}{\sqrt{1 - v^2/c^2}} (\vec{t} + (v/c^2) \vec{x})
\]

\[
x = \frac{1}{\sqrt{1 - v^2/c^2}} (v\vec{t} + \vec{x}).
\]

In matrix form this is

\[
\begin{bmatrix} t \\ x \end{bmatrix} = \frac{1}{\sqrt{1 - v^2/c^2}} \begin{bmatrix} 1 & v/c^2 \\ v & 1 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{x} \end{bmatrix}.
\]

(6)

**Physical Implications:**

Suppose that velocity \(v \ll c\), i.e., \(v\) is small relative to the speed of light. Then \(v/c\) is negligible and the Lorentz transformations in (6) reduce to

\[
\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{x} \end{bmatrix}
\]

or equivalently

\(t = \vec{t}\) and \(x = v \vec{t} + \vec{x}\).

These are the *Galilean transformations* of classical physics in which relativistic effects are not apparent. However, when relative speeds \(v\) are near \(c\), the Lorentz transformations produce some surprising and dramatic relativistic effects.

**Length Contraction:**

A spaceship flying through space along a line with constant speed \(v\) flies by the international space station. At instant \(\vec{t}\) in the moving frame \(\vec{K}(ct, \vec{x})\) of the spaceship, the ship’s captain observes that the endpoints of the space station are positioned at \(\vec{x}_1\) and \(\vec{x}_2\) on the \(\vec{x}\)-axis; thus its ordinary length measured by the ship’s captain is \(\Delta \vec{x} = \vec{x}_2 - \vec{x}_1\). Thinking of these measurements as events, their \(\vec{K}\)-coordinates are \((\vec{t}, \vec{x}_1)\) and \((\vec{t}, \vec{x}_2)\), and we can use equation (6) to change coordinates and calculate the ordinary length \(\Delta x = x_2 - x_1\) in the fixed reference frame \(K(ct, x)\) of the space station. According to (6), the relationship between the lengths \(\Delta \vec{x}\) and \(\Delta x\) at instant \(\vec{t}\) is

\[
\Delta x = x_2 - x_1 = \frac{v \vec{t} + \vec{x}_2}{\sqrt{1 - v^2/c^2}} - \frac{v \vec{t} + \vec{x}_1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - v^2/c^2}} \Delta \vec{x},
\]

or equivalently,

\[
\Delta \vec{x} = \sqrt{1 - v^2/c^2} \Delta x.
\]
Since $\sqrt{1-v^2/c^2} < 1$, ordinary length in frame $K$ appears to contract when viewed from from $\tilde{K}$. For example, if $v = .73c$, then

$$\sqrt{1-.73^2} \approx \sqrt{.47} \approx .69;$$

if $\Delta x = \sqrt{34} \approx 5.83$, then

$$\Delta \tilde{x} \approx (.69)(5.83) \approx 4.$$

So the ordinary length of the space station measured by the spaceship captain appears to be about 31% less than the ordinary length measured by the space station manager.

Figure 2. Length contracts as imaginary hyperbolic rotation angle increases.

In summary, to an observer in a reference frame moving along a straight line with constant speed $v$ relative to a fixed reference frame, the ordinary length of an object at rest in the fixed frame appears to be shorter than it does to an observer in the fixed frame by a factor of $\sqrt{1-v^2/c^2}$. And indeed, $\Delta \tilde{x} \to 0$ as $v \to c$. This phenomenon is called the Lorentz length contraction.

**Time Dilation:**

Now suppose a clock on board the spaceship is positioned at the origin $\tilde{O}$ in the moving frame $\tilde{K}$ of the spaceship. As the spaceship passes the space station, the captain takes two clock readings $\tilde{t}_1$ and $\tilde{t}_2$ and determines the elapsed time to be $\Delta \tilde{t} = \tilde{t}_2 - \tilde{t}_1$. Thinking of these two readings as events, their $\tilde{K}$-coordinates are $(\tilde{t}_1, 0)$ and $(\tilde{t}_2, 0)$, and the relationship between the elapsed time $\Delta \tilde{t}$ measured in the moving frame and the elapsed time $\Delta t$ in the fixed frame given by (6) is

$$\Delta t = \frac{\tilde{t}_2}{\sqrt{1-v^2/c^2}} - \frac{\tilde{t}_1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-v^2/c^2}} \Delta \tilde{t}.$$

Since $\frac{1}{\sqrt{1-v^2/c^2}} > 1$, elapsed time in frame $\tilde{K}$ appears to dilate when viewed from $K$. For example, if $v = .73c$, then

$$\frac{1}{\sqrt{1-.73^2}} \approx \frac{1}{\sqrt{.47}} \approx 1.46;$$

if $\Delta \tilde{t} = 4$, then

$$\Delta t \approx (1.46)(4) \approx 5.84.$$

So as far as the space station manager can tell, space station clocks appear to run about 46% faster than clocks on board the passing spaceship.
In summary, to an observer in a fixed reference frame, the elapsed time measured in a reference frame moving along a straight line with constant speed \( v \) appears to dilate by a factor of \( \frac{1}{\sqrt{1-v^2/c^2}} \). And indeed, \( \Delta t \to \infty \) as \( v \to c \). This phenomenon is called the Lorentz time dilation.

**Moral:** Live fast; live long (relatively speaking...)!

**Exercises**

8. Consider a particle \( P \) positioned at the origin \( O \) in a frame \( \bar{K} \) moving relative to a fixed frame \( K \) in the positive \( x \)-direction. Prove that the world line of \( P \) in \( K \) lies inside the light cone.

9. Compute the factors of length contraction and time dilation when \( v = .9c \) and \( v = .99c \).

11-18-2010