Computational Application of a Transfer Algorithm to the Borromean Rings

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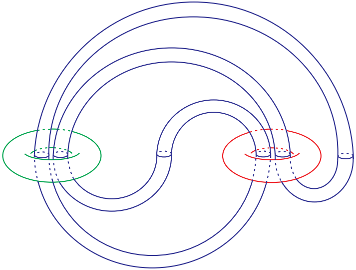
Advised by Ron Umble

Background



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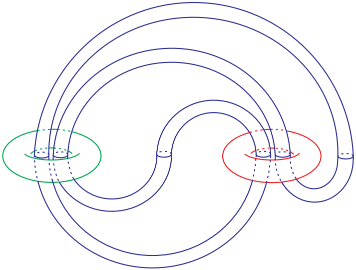

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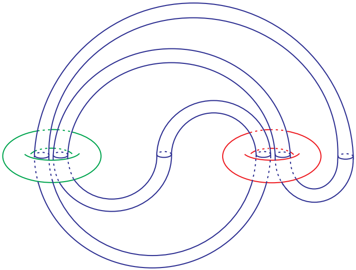

- started as summer (2015) project proposed by Ron Umble
- algorithm to possibly detect linkage in Brunnian links
- requires some algebraic topology and probably a lot of computation

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  - union of all cells is X

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  - a derivation of Cartesian product

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- $H_*(C) = \ker\partial / \mathrm{Im}\partial$
- Elements of $H_*(C)$ are cosets $[c] = c + \mathrm{Im}\partial$
- equivalence classes of nonbounding cycles that differ only by a boundary
- Note: Homology alone **does not detect linkage**!

Diagonal Approximation

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3. ∂ is a coderivation of Δ, i.e., $\Delta\partial = (\partial \times \mathrm{Id} + \mathrm{Id} \times \partial)\Delta$

Coproduct Notation

Some notation:

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- Higher coproducts on chains will be denoted by subscripts, e.g., $\Delta_3, \Delta_4, \ldots$
- Coproducts transferred to homology will be denoted by superscripts, e.g., $\Delta^2, \Delta^3, \ldots$

Brunnian Links

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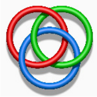
- Example: Borromean rings (3-component Brunnian link)



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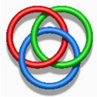

- We will denote the link complement in S^3 of an n-component Brunnian link by BR_n, $n \geq 3$.

Conjecture

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A diagonal approximation Δ_2 on $C(BR_n)$ induces

- *a primitive diagonal $\Delta^2 : H(BR_n) \otimes H(BR_n)$,*
- *trivial k-ary operations $\Delta^k : H(BR_n)^{\otimes k}$ for $3 \leq k < n$, and*
- *a non-trivial n-ary operation $\Delta^n : H(BR) \to H(BR)^{\otimes n}$.*

Predictions

Hence, for the Borromean rings we are expected to find:

- a primitive Δ^2
- a non-trivial Δ^3

These coproducts will be induced through the Transfer Algorithm.

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Transferring Coproducts

Goal:

$$\begin{array}{c} A_\infty\text{-coalgebra on chains} \\ (C, \partial, \Delta_2, \Delta_3, \ldots) \\ \downarrow \\ (H, 0, \Delta^2, \Delta^3, \ldots) \\ A_\infty\text{-coalgebra in homology} \end{array}$$

Transferring Coproducts

Required input:

- Coalgebra on chains $(C, \partial, \Delta_2, \Delta_3, \ldots)$ and
- a cycle-selecting map $g : H \to Z(C)$, where $Z(C)$ denotes the subspace of cycles in C.

Note: In practice we only required Δ_2 at the outset and computed the rest as needed.

How Does It Work?

Strategy: Construct a chain map from the top dimension and codim-1 cells of the $(n-1)$-dimensional multiplihedron, denoted J_n, to maps between H and $C^{\otimes n}$.

Beginning Steps

- J_n is a polytope that captures the combinatiorial structure of mapping between two A_∞-coalgebras.

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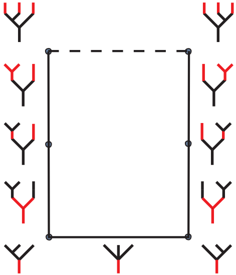
- Consider J_1 and J_2.

Extending to J_3



$$\mapsto$$

$$g^{\otimes 3}\left(\Delta^2 \otimes \mathbf{1}\right)\Delta^2 \qquad g^{\otimes 3}\left(\mathbf{1} \otimes \Delta^2\right)\Delta^2$$

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Good News

Linear algebra provides robust and theoretically correct methods for solving the various induction steps of the transfer algorithm.

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Bad News

The matrices are too large to be solved within a reasonable amount of storage space and time.

Two Problems

Problem (Preboundary)

Given a cycle $x \in C^{\otimes n}$ *of degree* k*, find a chain* $y \in C^{\otimes n}$ *of degree* $k+1$*, such that* $\partial(y) = x$.

Problem (Factorization)

Given a cycle $c \in Z(C^{\otimes n})$*, find all subcycles of* c *of the form* $Z(C)^{\otimes n}$.

Preboundary Problem: Δ_3

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- First problem arose in computing Δ_3
- It is the preboundary of $(\Delta_2 \otimes 1 + 1 \otimes \Delta_2)\Delta_2$
- Brute force linear algebra approach entails 1.8 mil row $\times$ 4 mil column matrix
- Instead, solved with a best-first search algorithm

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- Hence, non-boundary cycles in ϕ_n in should be of the form $Z(C)^{\otimes(n+2)}$
- Again, an algorithmic approach appears to be a feasible alternative

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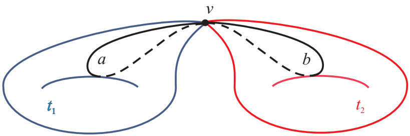
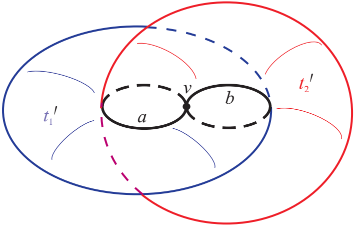
Unlink vs. Hopf Link



2-Component Unlink



Hopf Link

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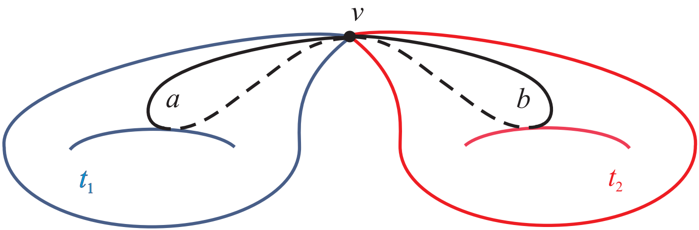

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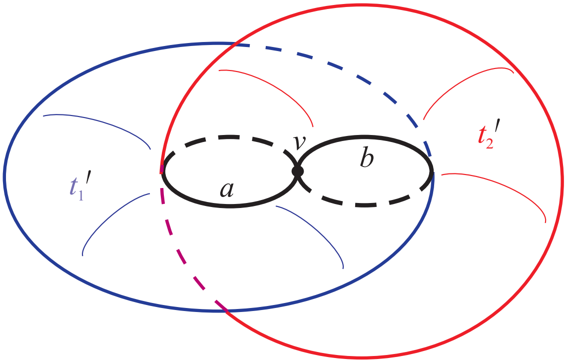

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Results

Transfer algorithm yields

- a primitive Δ^2 for unlink

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- all of which are consistent with the conjecture!

Future Work

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    - the last steps of BR_3 were actually done by hand, and BR_4 will only be worse
- both the preboundary and factorization algorithms need improvement

Thank You

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