An A_∞-Coalgebra Structure on a Polygon

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Abstract

Let P be a polygon with n vertices, let V be the graded vector space generated by the vertices, edges, and region of P, and let $\partial : V \to V$ be the map induced by the geometric boundary. There is a homotopy coassociative coproduct $\Delta_2 : V \to V \otimes V$, a *coassociator* $\Delta_3 : V \to V^{\otimes 3}$, and non-vanishing higher order operations $\Delta_k : V \to V^{\otimes k}$ for all $k < n$. The vector space V together with ∂ and the operations $\{\Delta_k\}$ is an *A_∞-coalgebra.* To our knowledge, this project presents the first such family of examples.

The Associahedra

Given a sequence of n objects, how many different ways are there to associate them? When $n = 3$ there are two possibilities: •(••) and (••)•, when $n = 4$ there are five possibilities: ((••)•)•, (•(••))•, (••)(••), •((••)•), and •(•(••)), and so on. The associations of n objects are encoded combinatorially by the associahedron K_n, which is an $(n-2)$-dimensional polytope: K_2 is a point, K_3 is a closed interval, K_4 is a pentagon, and so on. The top $(n-2)$-dimensional cell of K_n corresponds to the initial sequence of unparenthesized objects; the $(n-3)$-dimensional cells correspond to the various ways to insert one pair of parentheses; the $(n-4)$-dimensional cells correspond to the various ways to insert two pairs of parentheses, and so on until you reach the vertices in dimension 0. For example, the 2-dimensional region of the pentagon K_4 corresponds to four unparenthesized objects, each edge corresponds to one of the five ways to insert one pair of parentheses, and each vertex corresponds to one of the five ways to insert a second pair of parentheses (see Figure 1). In our application, the associahedra organize the data that defines an A_∞-coalgebra.

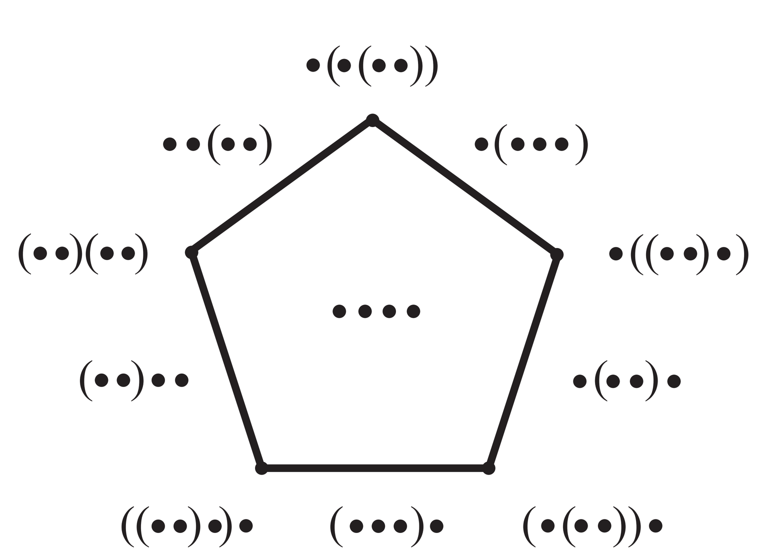

Figure 1: The associahedron K_4.

Differential Graded Vector Spaces

Consider an n-gon P with edges $e_1, e_2, ..., e_n$ and vertices $v_1, v_2, ..., v_n$, labeled so that v_i and v_{i+1} are the endpoints of e_i when $i < n$. The vector space generated by P, its vertices, and its edges is a *graded* vector space denoted by $C_*(P)$. Each basis vector has a geometric dimension: $\dim P = 2$, $\dim e_i = 1$, and $\dim v_j = 0$. The vectors in $C_*(P)$ are called the *cellular chains of* P (see Figure 2). The geometric boundary induces a linear operator ∂ on $C_*(P)$: $\partial v_i = 0$, $\partial e_i = v_i + v_{i+1}$ for $i < n$, $\partial e_n = v_1 + v_n$, and $\partial P = e_1 + e_2 + \cdots + e_n$. Notice that $\partial \circ \partial = 0$ because the boundary of a vertex is empty, the boundary of an edge is two vertices, and the boundary of P is a simple closed curve, all of which have empty boundary. The pair $(C_*(P), \partial)$ is called a *differential graded vector space (dgvs).* Here we use $\mathbb{Z}_2$ coefficients, but our results hold over any field.

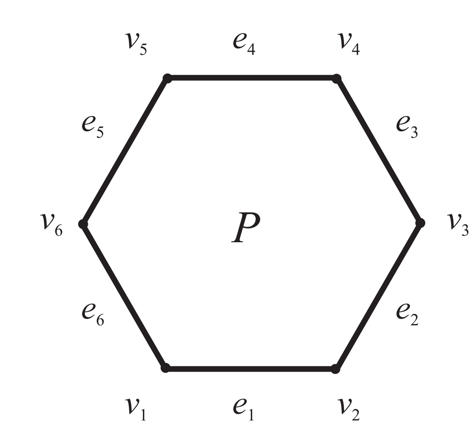

Figure 2: The basis for $C_*(P)$ when $n = 6$.

The Vector Space $Hom_*(V, V^{\otimes n})$

Given a dgvs (V, ∂), the n-fold tensor product $V^{\otimes n}$ is a dgvs with differential $\partial^{\otimes}$ defined by

$$\partial^{\otimes}(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n} x_1 \otimes \cdots \otimes \partial x_i \otimes \cdots \otimes x_n.$$

The *degree* of a linear map $f : V \to V^{\otimes n}$ is the number of dimensions f raises a vector in V. For example, $\deg \partial = -1$ because ∂ lowers dimension by 1. Let $Hom_p(V, V^{\otimes n})$ denote the set of all linear maps $f : V \to V^{\otimes n}$ of degree p. Then $Hom_*(V, V^{\otimes n})$ is a dgvs with differntial δ defined by

$$\delta(f) = f \circ \partial + \partial^{\otimes} \circ f.$$

Let $\mathbf{1} : V \to V$ be the identity map. For each $n \geq 2$, choose a map $\Delta_n \in Hom_{n-2}(V, V^{\otimes n})$ and think of a parenthesization of n objects as a composition of these maps. For example,

$$\bullet(\bullet\bullet\bullet) \leftrightarrow (\mathbf{1} \otimes \Delta_3)\,\Delta_2.$$

A_∞-coalgebras Defined

For $2 < k \leq n-2$, let $\alpha : C_{n-k}(K_n) \to Hom_{n-k}(V, V^{\otimes n})$ denote the correspondence between cells of K_n indexed by parenthesizations of n objects and compositions of maps as above. Linearly extend α to all of $C_*(K_n)$. Let θ_n denote the $(n-2)$-dimensional cell of K_n.

Definition. *A dgvs (V, ∂) together with a family of maps $\{\Delta_2, \Delta_3, \ldots\}$ is an A_∞* **-coalgebra** *iff for all $n \geq 2$,*

$$\alpha\partial(\theta_n) = \delta\alpha(\theta_n).$$

Note that ∂K_n is the sum of those cells in K_n representing n objects with one pair of inserted parentheses. The map α identifies these cells with all possible quadratic compositions of the maps $\{\mathbf{1}, \Delta_2, \Delta_3, \ldots\}$ in $Hom_*(V, V^{\otimes n})$. Thus (V, ∂) is an A_∞-coalgebra iff the following relation holds for each $n \geq 2$:

$$\delta(\Delta_n) = \sum_{l=1}^{n-2} \sum_{i=0}^{n-l-1} \left(1^{\otimes i} \otimes \Delta_{l+1} \otimes 1^{\otimes n-l-i-1}\right) \Delta_{n-l}.$$

Main Result

Given an n-gon P, $n \geq 3$, define

$$\begin{aligned}
\Delta_2(v_i) &= v_i \otimes v_i \\
\Delta_2(e_i) &= v_i \otimes e_i + e_i \otimes v_{i+1} \text{ for } i < n \\
\Delta_2(e_n) &= v_1 \otimes e_n + e_n \otimes v_n \\
\Delta_2(P) &= v_1 \otimes P + P \otimes v_n + \sum_{0<i_1<i_2<n} e_{i_1} \otimes e_{i_2} \\
\Delta_k(P) &= \sum_{0<i_1<i_2<\cdots<i_k<n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \text{ for } k > 2 \\
\Delta_k(\sigma) &= 0 \text{ for } \sigma \neq P.
\end{aligned}$$

Theorem. *Let ∂ be the differential operator on $C_*(P)$ induced by the geometric boundary. Then $(C_*(P), \partial, \Delta_2, \Delta_3, \ldots)$ is an A_∞-coalgebra. Moreovoer, $\Delta_k \neq 0$ for all $k < n$ and vanishes for all $k \geq n$.*

Open Questions

It would be interesting to know whether or not the theorem above can be generalized to more complicated objects such as 3-dimensional polyhedra and higher dimensional polytopes. Future work could begin with a cube, consider general 3-dimensional solids, and ultimately consider higher dimensional polytopes.