An A_∞-coalgebra Structure on the Cellular Chains of a Polygon

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Introduction

- Suppose you had several objects that you wanted to combine using a binary operation.

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- What are all the different ways that you could associate them?

Introduction

Example

Three objects can be associated in two ways:

- $(\bullet\bullet)\bullet$
- $\bullet(\bullet\bullet)$

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Four objects can be associated in five ways:

- $((\bullet\bullet)\bullet)\bullet$
- $(\bullet(\bullet\bullet))\bullet$
- $(\bullet\bullet)(\bullet\bullet)$
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Definition of an Associahedron

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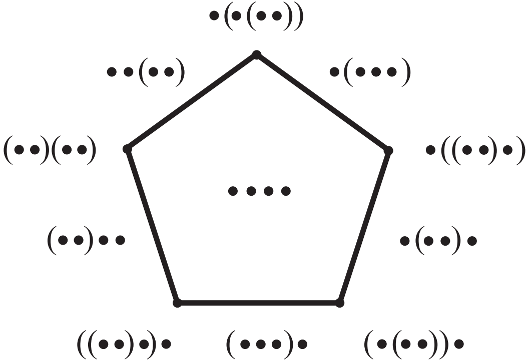
Example: The associahedron K_3.

Another Associahedron Example



Example: The associahedron K_4.

Definition of a Differential Graded Vector Space

Definition

A *differential graded (d.g.) vector space* is a vector space $V = \oplus_{i \geq 0} V_i$ equipped with a differential operator $\partial : V_* \to V_{*-1}$ such that $\partial \circ \partial = 0$.

Example of a Differential Graded Vector Space

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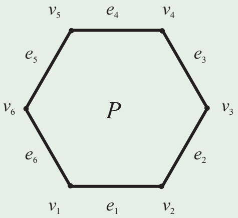

An n-gon for $n = 6$

- $C_0(P)$ is the vector space generated by the vertices
- $C_1(P)$ is the vector space generated by the edges
- $C_2(P)$ is the vector space generated by the single face P
- $C_*(P) = C_0(P) \oplus C_1(P) \oplus C_2(P)$ is called the *cellular chains* of P

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- In order to properly define ∂, we first choose a *initial* vertex and a *terminal* vertex. In this example, we will say that v_1 is the initial vertex and that v_6 is the terminal vertex (in general, the last vertex will be the terminal vertex).

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- This defines a poset in which all verticies along each side of the n-gon form an increasing sequence from the initial vertex to the terminal vertex. Next we orient each edge so that it is pointing away from the initial vertex and towards the terminal vertex

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- This defines a poset in which all verticies along each side of the n-gon form an increasing sequence from the initial vertex to the terminal vertex. Next we orient each edge so that it is pointing away from the initial vertex and towards the terminal vertex
- Finally, we choose to orient the polygon counterclockwise, and assign a sign to each edge based on whether or not it goes "against" the orientation

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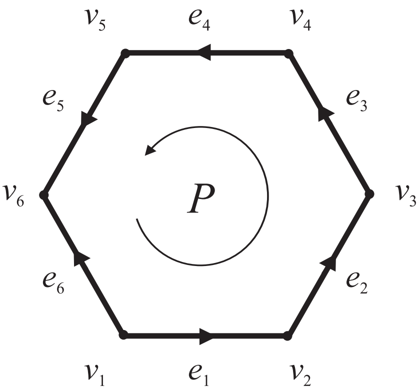

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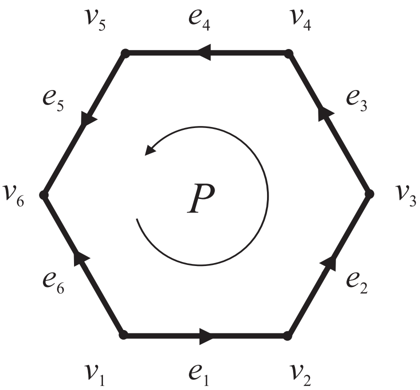

Define ∂ as the geometric boundary

- $\partial v_i = 0$
- $\partial e_i = v_{i+1} - v_i$ for $i \neq 6$ and $\partial e_6 = v_6 - v_1$
- $\partial P = e_1 + e_2 + e_3 + e_4 + e_5 - e_6$

Chain Maps

Definition

Let (V, ∂_V) and (W, ∂_W) be a d.g. vector spaces. A linear map $f : V \to W$ has *degree* p if $f : V_i \to W_{i+p}$; the map f is a *chain map of degree* p if $\partial_W \circ f = (-1)^p f \circ \partial_V$.

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Theorem

$Hom_*(V, W)$ is a d.g. vector space with differential defined by $\delta(f) = f \circ \partial_V - (-1)^p \partial_W \circ f$.

Differential Graded Coalgebras

Definition

A d.g. coalgebra is a d.g. vector space (V, ∂) together with a coassociative coproduct $\Delta_2 : V \to V \otimes V$ of degree 0 such that ∂ is a coderivation of Δ_2, i.e. $\Delta_2 \partial = (\partial \otimes \mathbf{1} + \mathbf{1} \otimes \partial)\, \Delta_2$.

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Note that if Δ_2 is coassociative, then we would have that $(\Delta_2 \otimes \mathbf{1})\, \Delta_2 = (\mathbf{1} \otimes \Delta_2)\, \Delta_2$

Coassociation up to Homotopy

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Definition

Define Δ_3 (if possible), in such a way so that
$\delta(\Delta_3) = (\Delta_2 \otimes \mathbf{1})\,\Delta_2 - (\mathbf{1} \otimes \Delta_2)\,\Delta_2$

Coassociation up to Homotopy

Then Δ_3 is second order coassociative if

$$\begin{aligned}(\Delta_3 \otimes \mathbf{1})\,\Delta_2 + (\mathbf{1} \otimes \Delta_3)\,\Delta_2 - (\Delta_2 \otimes \mathbf{1} \otimes \mathbf{1})\,\Delta_3 \\ + (\mathbf{1} \otimes \Delta_2 \otimes \mathbf{1})\,\Delta_3 - (\mathbf{1} \otimes \mathbf{1} \otimes \Delta_2)\,\Delta_3 = 0\end{aligned}$$

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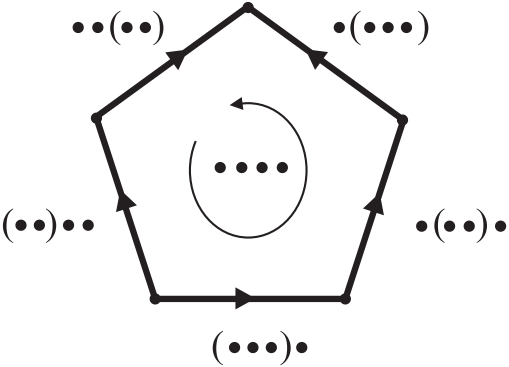

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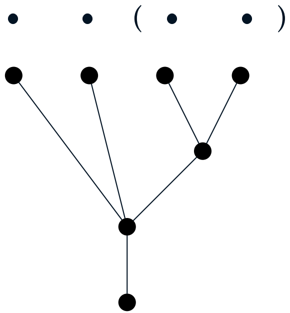

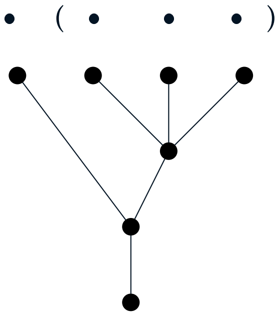

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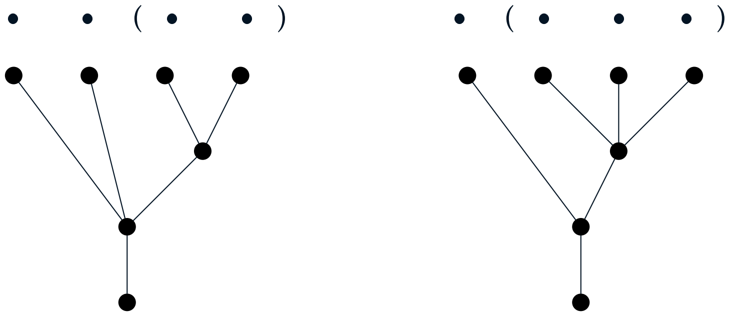

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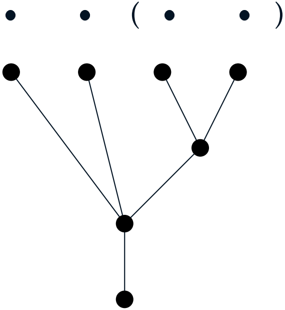

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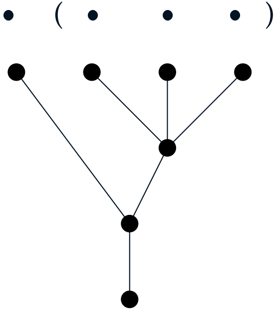

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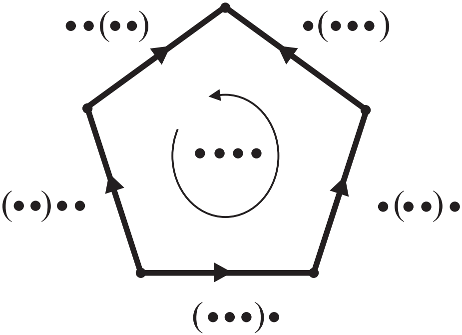

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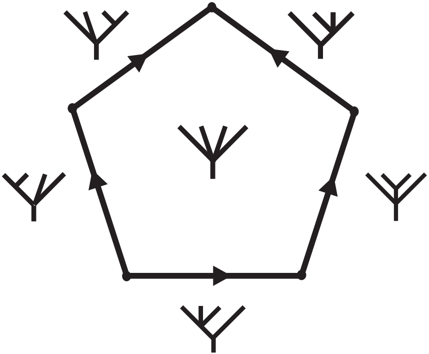

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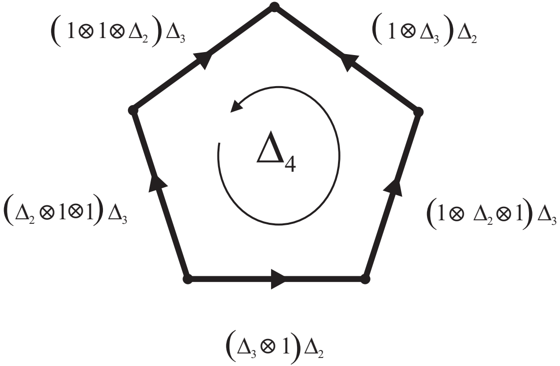

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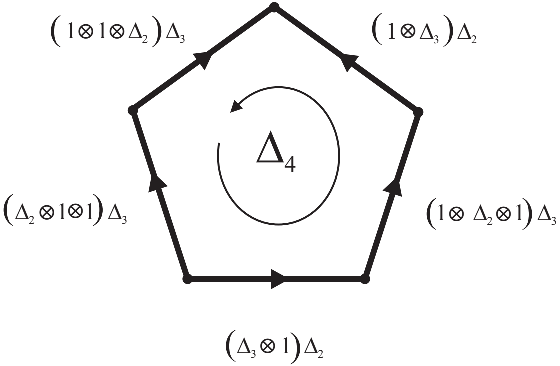

$$(\Delta_3 \otimes \mathbf{1})\,\Delta_2 + (\mathbf{1} \otimes \Delta_3)\,\Delta_2 - (\Delta_2 \otimes \mathbf{1} \otimes \mathbf{1})\,\Delta_3 + (\mathbf{1} \otimes \Delta_2 \otimes \mathbf{1})\,\Delta_3 - (\mathbf{1} \otimes \mathbf{1} \otimes \Delta_2)\,\Delta_3 = 0$$

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- If Δ_4 is not third order coassociative, then perhaps we can find a Δ_5, and then maybe a Δ_6, and so on as long as necessary

Definition of an A_∞-coalgebra

Definition

$(V, \partial, \Delta_2, \Delta_3, \ldots)$ is an A_∞-coalgebra if for all $k \geq 2$,

$$\delta\left(\Delta_k\right) = \sum_{l=1}^{k-2} \sum_{i=0}^{k-l-1} (-1)^{l(k+i+1)} \left(1^{\otimes i} \otimes \Delta_{l+1} \otimes 1^{\otimes k-l-i-1}\right) \Delta_{k-l}. \tag{1}$$

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- Today, we will present the first known example of an A_∞-coalgebra which has a finite number of non-vanishing Δ_k.

Main Result

Let P be a counterclockwise oriented polygon with n sides where $n \geq 3$. Label the vertices $v_1, v_2, \ldots v_n$ and the edges $e_1, e_2, \ldots e_n$ as in the following diagram:

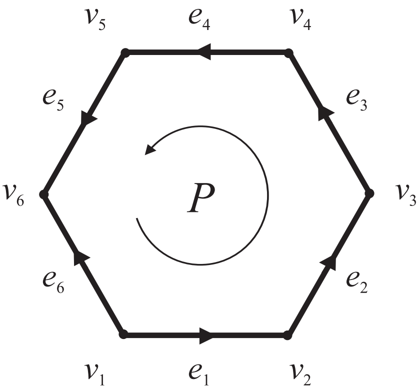

Figure 3: An n-gon for $n = 6$

Main Result

Define the diagonal Δ_2 by:

$$\Delta_2(P) = v_1 \otimes P + P \otimes v_n + \sum_{0<i_1<i_2<n} e_{i_1} \otimes e_{i_2},$$
$$\Delta_2(e_i) = v_i \otimes e_i + e_i \otimes v_{i+1} \text{ if } i < n,$$
$$\Delta_2(e_n) = v_1 \otimes e_n + e_n \otimes v_n,$$
$$\Delta_2(v_i) = v_i \otimes v_i,$$

and define the k-ary A_∞-coalgebra operations Δ_k where $k > 2$ as:

$$\Delta_k(P) = \sum_{0<i_1<i_2<\cdots<i_k<n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$$
$$\Delta_k(\sigma) = 0 \text{ when } \sigma \neq P.$$

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Example

Verify the relation for Δ_2 on e_i for $i < n$
For $k = 2$, relation (1) simplifies to: $\Delta_2 \partial - \partial^{\otimes 2} \Delta_2 = 0$ Then,

$$
\begin{aligned}
&(\Delta_2 \partial - \partial^{\otimes 2} \Delta_2)(e_i) \\
=&\Delta_2 \partial(e_i) - \partial^{\otimes 2} \Delta_2(e_i) \\
=&\Delta_2(v_{i+1} - v_i) - (\partial \otimes \mathbf{1} + \mathbf{1} \otimes \partial)(v_i \otimes e_i + e_i \otimes v_{i+1}) \\
=&v_{i+1} \otimes v_{i+1} - v_i \otimes v_i - 0 \otimes e_i - v_i \otimes (v_{i+1} - v_i) \\
&- (v_{i+1} - v_i) \otimes v_{i+1} - e_i \otimes 0 = 0
\end{aligned}
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Let $\Delta_5 \Delta_3(P)$ denote
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- Since all Δ_k for $k > 2$ vanish when applied to a vertex or edge, relation (1) only needs to be verified on P.
- Additionally, because of the way that higher Δ_k's vanish when applied to anything but P, any sequence in relation (1) of the form $\Delta_j \Delta_i$ where $j, i > 2$ vanish.

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Example

$\Delta_5\Delta_3(P)$ vanishes.

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Proof of Main Result

Recall relation (1):

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which can be written as:

$$\sum_{i=0}^{k-1} \left(1^{\otimes i} \otimes \partial \otimes 1^{\otimes k-i-1}\right) \Delta_k - (-1)^k \Delta_k \partial$$

$$= \sum_{l=1}^{k-2}\sum_{i=0}^{k-l-1} (-1)^{l(k+i+1)} \left(1^{\otimes i} \otimes \Delta_{l+1} \otimes 1^{\otimes k-l-i-1}\right) \Delta_{k-l}.$$

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Which can be simplified to

$$(\partial \Delta_k + \Delta_{k-1}\Delta_2 + \Delta_2\Delta_{k-1})(P) = 0 \tag{2}$$

Main Idea

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- In order to verify relation (2), we will simply examine each of the three sets of operations and ask what type of terms they produce when applied to P
- We can then note that any terms generated by one sequence of operations cancel with terms generated by another sequence of operations.

Classifying Types of Terms

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Example

Some terms that can be generated:

- $e_3 \otimes e_7 \otimes v_{10} \otimes e_{13}$
- $v_1 \otimes e_2 \otimes e_4 \otimes e_5 \otimes e_6$
- $e_4 \otimes v_7 \otimes e_7$
- $e_1 \otimes v_2 \otimes e_6 \otimes e_8 \otimes e_9 \otimes e_{17}$

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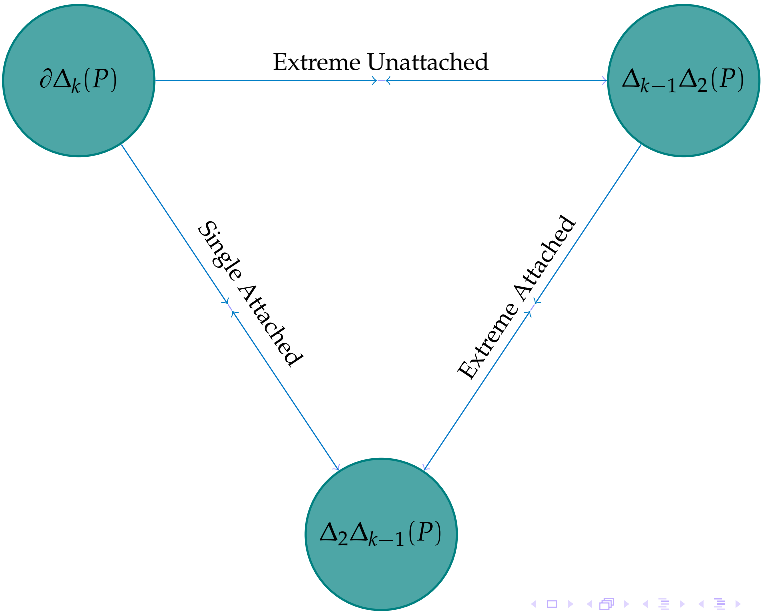
- $\Delta_2\Delta_{k-1}(P)$: Produces all possible singly attached terms (both extreme and non extreme)

Canceling Terms



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- Thankfully however, we can show that $\Delta_2\Delta_2(P)$ "behaves" the same way as $\Delta_{k-1}\Delta_2(P)$ and $\Delta_2\Delta_{k-1}(P)$ together (that is, produces all the terms that would have normally been left over after combining the terms produced by $\Delta_{k-1}\Delta_2(P)$ and $\Delta_2\Delta_{k-1}(P)$).

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- Therefore, we have verified that we have an A_∞-coalgebra!

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Using this theorem, we can extend our A_∞-coalgebra to allow some other vertex to be terminal instead of v_n. Suppose that v_t is the new terminal vertex. Then define:

$$\Delta'_2(P) = v_1 \otimes P + P \otimes v_t + \sum_{0<i_1<i_2<t} (e_{i_1} \otimes e_{i_2}) - \sum_{n \geq i_1 > i_2 \geq t} (e_{i_1} \otimes e_{i_2})$$

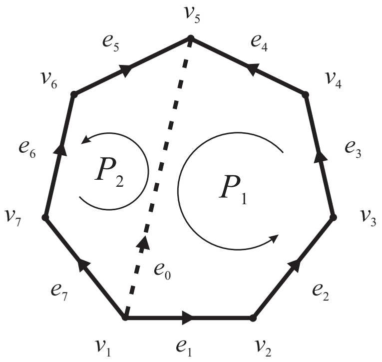
and for $k > 2$

$$\Delta'_k(P) = \sum_{0<i_1<i_2<\cdots<i_k<t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} - \sum_{n \geq i_1 > i_2 > \cdots > i_k \geq t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \text{ and}$$

$$\Delta'_k(\sigma) = 0 \text{ when } \sigma \neq P.$$

Corollary

To prove this, we divide our polygon, and use our original result on both parts.



Example on a 7-gon with terminal vertex v_5

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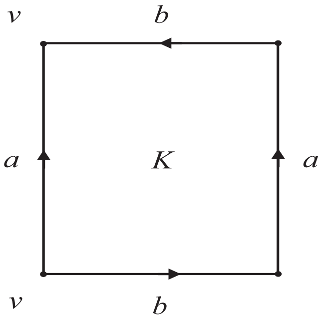
The operation $\{\Delta'_n\}_{n\geq 2}$ *defined above satisfies all* A_∞*-coalgebra relations on cellular chains of* P *where the initial vertex is* v_1 *and the terminal vertex is* v_t *where* $1 < t \leq n$*. Furthermore, all* Δ_k *for* $k < n$ *are non-trivial, and all* Δ_k *for* $k \geq max\{t, n-t+2\}$ *vanish.*

An Application

Using our result, we obtain a higher-order coalgebra structure on the cellular chains of the Klein bottle!



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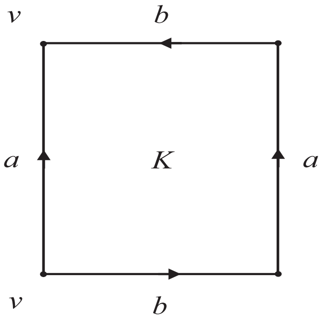

- $\Delta_2 K = K \otimes v + v \otimes K + a \otimes b + b \otimes b + b \otimes a$
- $\Delta_3 K = b \otimes a \otimes b.$

Acknowledgements

- I would like to thank my thesis adviser, Dr. Umble for introducing me to the topic, for his help, and all his advice.
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Thanks!