An A_∞-coalgebra Structure on the Cellular Chains of a Polygon

A Senior Thesis Submitted to the Department of Mathematics In Partial Fulfillment of the Requirements for the Departmental Honors Baccalaureate

By Quinn Minnich

Millersville, Pennsylvania

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This Senior Thesis was completed in the Department of Mathematics, defended before and approved by the following members of the Thesis committee

Ronald N. Umble, Ph.D (Thesis Advisor)
Professor of Mathematics
Millersille University

Zhigang Han, Ph.D
Associate Professor of Mathematics
Millersville University

Bruce M. Ikenaga, Ph.D
Associate Professor of Mathematics
Millersville University

Neal W. Stoltzfus, Ph.D
Professor of Mathmatics Emeritus
Louisiana State University

Abstract

Let P be a polygon with n vertices, let V be the graded vector space generated by the vertices, edges, and region of P, and let $\partial : V \to V$ be the map induced by the geometric boundary. There is a homotopy coassociative coproduct $\Delta_2 : V \to V \otimes V$, a *coassociator* $\Delta_3 : V \to V^{\otimes 3}$, and non-vanishing higher order operations $\Delta_k : V \to V^{\otimes k}$ for all $k < n$. The vector space V together with ∂ and the operations $\{\Delta_k\}$ is an A_∞-coalgebra. To our knowledge, this project presents the first such family of examples.

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1 Introduction

This thesis presents an example of an A_∞-coalgebra structure on an n-gon. As motivation, we first define the notion of homotopy associativity, a family of polytopes called associahedra, and the dual notion of an A_∞-algebra, given by J. Stasheff [5] in the setting of base pointed loop spaces.

Let S be a surface embedded in $\mathbb{R}^3$ and let $*$ be a base point on S. A *base pointed loop on* S is a continuous map $\alpha : I \to S$ such that $\alpha(0) = \alpha(1) = *$. Let ΩS denote the set of all base pointed loops on S. Given $\alpha, \beta \in \Omega S$, define their *product* $\alpha \cdot \beta \in \Omega S$ to be

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

A *homotopy from* α *to* β is a continuous map $H : I \to \Omega S$ such that $H(0) = \alpha$ and $H(1) = \beta$. Thus $\{H(s) : s \in I\}$ is a 1-parameter family of loops that continuously deforms α to β.

Let $\alpha, \beta, \gamma \in \Omega S$. Although $(\alpha \cdot \beta) \cdot \gamma \neq \alpha \cdot (\beta \cdot \gamma)$, the loops $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ are homotopic via a linear change of parameter H that decreases the speed of α from quadruple to double speed and increases the speed of γ from double to quadruple speed as indicated by the following diagram:

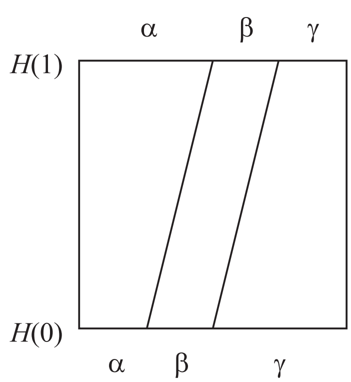

Figure 1: A homotopy H between the two associations of three loops.

Thus loop multiplication is *homotopy associative.*

Let $m_2(\alpha \otimes \beta) = \alpha \cdot \beta$; then $m_2 : \Omega S \otimes \Omega S \to \Omega S$. Let $\mathbf{1} : \Omega S \to \Omega S$ be the identity map; then $(\alpha \cdot \beta) \cdot \gamma = m_2(m_2 \otimes \mathbf{1})(\alpha \otimes \beta \otimes \gamma)$ and $\alpha \cdot (\beta \cdot \gamma) = m_2(\mathbf{1} \otimes m_2)(\alpha \otimes \beta \otimes \gamma)$. Consider

$m_2(m_2 \otimes \mathbf{1}), m_2(\mathbf{1} \otimes m_2) : \Omega S^{\otimes 3} \to \Omega S$ and think of the homotopy H from $(\alpha \cdot \beta) \cdot \gamma$ to $\alpha \cdot (\beta \cdot \gamma)$ as a 3-ary operation $m_3 : \Omega S^{\otimes 3} \to \Omega S$. Identify m_3 with the interval $[0,1]$; identify the endpoint 0 with $m_2(m_2 \otimes \mathbf{1})$ and the endpoint 1 with $m_2(\mathbf{1} \otimes m_2)$. Then the boundary $\partial m_3 = m_2(\mathbf{1} \otimes m_2) - m_2(m_2 \otimes \mathbf{1})$ and the parameter space $[0,1]$ identified with m_3 is called the associahedron K_3. The associahedron K_3 controls homotopy associativity in three variables.

In a similar way, homotopy associativity in four variables is controlled by a parameter space called the associahedron K_4. The vertices of K_4, which is a pentagon, are identified with the five ways one can parenthesize four variables, the edges of the pentagon are identified with the homotopies that preform a single shift of parentheses, and the region bounded by the pentagon is identified with a 4-ary operation $m_4 : \Omega S^{\otimes 4} \to \Omega S$. Then $\partial m_4 = m_2(m_3 \otimes \mathbf{1}) - m_3(m_2 \otimes \mathbf{1} \otimes \mathbf{1}) + m_3(\mathbf{1} \otimes m_2 \otimes \mathbf{1}) - m_3(\mathbf{1} \otimes \mathbf{1} \otimes m_2) + m_2(\mathbf{1} \otimes m_3)$. Homotopy associativity in n-variables is controlled by the associahedron K_n, which is an $n-2$ dimensional polytope whose vertices are identified with the various ways one can parenthesize n variables.

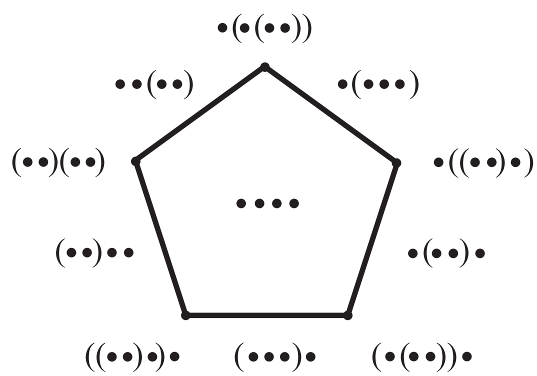

Figure 2: The associahedron K_4.

While associahedra are interesting geometric objects, they can also be used to form some interesting algebraic structures, including A_∞-coalgebras. The definition of an A_∞-coalgebra and how it relates to associahedra requires the definition of a differential graded vector space.

A *differential graded (d.g.) vector space* is a vector space $V = \oplus_{i \geq 0} V_i$ equipped with a differential operator $\partial : V_* \to V_{*-1}$ such that $\partial \circ \partial = 0$. For example, consider a polyhedron P such as a cube or a tetrahedron and think of its vertices, edges, faces, and 3-dimensional solid as i-dimensional basis vectors. Let $C_i(P)$ be the vector space generated by the i-dimensional basis vectors; then $C_*(P) = C_0(P) \oplus C_1(P) \oplus C_2(P) \oplus C_3(P)$ is a d.g. vector space whose differential operator ∂ is induced by the geometric boundary of P. The d.g. vector space $(C_*(P), \partial)$ is called the *cellular chains of* P.

Let (V, ∂_V) and (W, ∂_W) be d.g. vector spaces. A linear map $f : V \to W$ has *degree* p if $f : V_i \to W_{i+p}$; the map f is a *chain map of degree* p if $\partial_W \circ f = (-1)^p f \circ \partial_V$. Let $Hom_p(V, W)$ denote the vector space of all linear maps $f : V \to W$ of degree p.

Proposition 1 *$Hom_*(V, W)$ is a d.g. vector space with differential defined by $\delta(f) = f \circ \partial_V - (-1)^p \partial_W \circ f$. Where P is the degree of f*

Proof. To verify that $\delta(f)$ is in fact a differential, we must show that $\delta \circ \delta(f) = 0$. First of all, note that since f has degree P, $f \circ \partial_V$ and $\partial_W \circ f$ have degree $p-1$, and hence $\delta(f)$ has degree $p-1$. The proof now proceeds as follows:

$$\begin{aligned}
&\delta \circ \delta(f) \\
=&\delta\left(f \circ \partial_V - (-1)^p \partial_W \circ f\right) \\
=&\delta\left(f \circ \partial_V\right) - (-1)^p \delta\left(\partial_W \circ f\right) \\
=&\left((f \circ \partial_V) \circ \partial_V - (-1)^{p-1} \partial_W \circ (f \circ \partial_V)\right) \\
&\qquad - (-1)^p \left((\partial_W \circ f) \circ \partial_V - (-1)^{p-1} \partial_W \circ (\partial_W \circ f)\right) \\
=&-(-1)^{p-1}\left(\partial_W \circ f \circ \partial_V\right) - (-1)^p \left(\partial_W \circ f \circ \partial_V\right) = 0
\end{aligned}$$

■

Now if all of the functions in $Hom_*(V, W)$ were chain maps, the differential of each element of $Hom_*(V, W)$ would be zero. If the differential is not zero, then we can use the result to measure, in a sense, how far off the function is from being a chain map. Let $m_2 \in Hom_*\left(C_*(P)^{\otimes 2}, C_*(P)\right)$ and $m_3 \in Hom_*\left(C_*(P)^{\otimes 3}, C_*(P)\right)$. Then in the case of $f = m_3$, which has degree 1, we have

$$\delta(m_3) = m_3 \circ \partial^{\otimes 3} - (-1)^1 \partial \circ m_3 = \partial \circ m_3 = m_2(\mathbf{1} \otimes m_2) - m_2(m_2 \otimes \mathbf{1}),$$

where we use the fact that the boundary of a loop is 0. Then $\delta(m_3)$ measures the *deviation of* m_2 *from associativity*, and under certain circumstances we can express this deviation in terms of a chain map from the cellular chains $C_*(K_n)$ into $Hom_*\left(C_*(P)^{\otimes 3}, C_*(P)\right)$ and $f = m_3$.

Definition 2 *A d.g. algebra is a d.g. vector space (V, ∂) together with an associative product $m_2 : V \otimes V \to V$ of degree 0 such that ∂ is a derivation of m_2, i.e. $\partial m_2 = m_2(\partial \otimes \mathbf{1} + \mathbf{1} \otimes \partial)$.*

If m_2 fails to be associative, but is associative up to homotopy, there is a 3-ary chain homotopy $m_3 : V^{\otimes 3} \to V$ such that $\delta(m_3) = (m_2 \otimes \mathbf{1})\, m_2 - (\mathbf{1} \otimes m_2)\, m_2$. This 3-ary chain homotopy can still exist even if m_2 is associative, but it's required when m_2 is associative up to homotopy.

For each $n \geq 2$, let K_n denote the $(n-2)$-dimensional associahedron and consider its cellular chains $C_*(K_n)$. Choose a top dimensional generator $\theta_n \in C_{n-2}(K_n)$.

Definition 3 *Let (V, ∂) be a d.g. vector space. For each $n \geq 2$, choose a chain map $\alpha_n : C_*(K_n) \to Hom(V^{\otimes n}, V)$ of degree 0. Define $m_n := \alpha_n(\theta_n)$. Then $(V, \partial, m_2, m_3, \ldots)$ is an A_∞-algebra if for all $n \geq 2$,*

$$\delta(m_3) = \sum_{l=1}^{n-2} \sum_{i=0}^{n-l-1} (-1)^{l(i+1)} m_{n-l}(\mathbf{1}^{\otimes i} \otimes m_{l+1} \otimes \mathbf{1}^{\otimes n-l-i-1}).$$

The definitions of a d.g. coalgebra and an A_∞-coalgebra mirror the definitions of a d.g. algebra and an A_∞-algebra, and proceed as follows.

Definition 4 *A d.g. coalgebra is a d.g. vector space (V, ∂) together with a coassociative coproduct $\Delta_2 : V \to V \otimes V$ of degree 0 such that ∂ is a coderivation of Δ_2, i.e. $\Delta_2 \partial = \partial \Delta_2$.*

For each $n \geq 2$, let K_n denote the $(n-2)$-dimensional associahedron and consider its cellular chains $C_*(K_n)$. Choose a top dimensional generator $\theta_n \in C_{n-2}(K_n)$.

Definition 5 *Let (V, ∂) be a d.g vector space. For each $n \geq 2$, choose a chain map $\alpha : C_*(K_n) \to Hom(V, V^{\otimes n})$ of degree 0 and define $\Delta_n := \alpha_n(\theta_n)$.*

Then $(V, \partial, \Delta_2, \Delta_3, \ldots)$ is an A_∞-coalgebra if for all $n \geq 2$,

$$\delta(\Delta_n) = \sum_{l=1}^{n-2} \sum_{i=0}^{n-l-1} (-1)^{l(n+i+1)} \left(1^{\otimes i} \otimes \Delta_{l+1} \otimes 1^{\otimes n-l-i-1}\right) \Delta_{n-l}. \tag{1}$$

Since α_n is a chain map of degree 0, the map α_n commutes with differentials and we have

$$\delta\left(\Delta_n\right) = \delta\alpha_n\left(\theta_n\right) = \alpha_n\partial\left(\theta_n\right).$$

Thus the right-hand side of (1) is image of $\partial\left(\theta_n\right)$ under α_n, which is the sum of the compositions represented by the dimension $n-3$ cells in the boundary of K_n, and $(-1)^{l(n+i+1)}$ is the combinatorial sign derived by Sansblidze and Umble[4]. For low dimensional values of n we have

$$\begin{aligned}
&\Delta_2\partial - \partial\Delta_2 = 0\\
&\Delta_3\partial + \partial\Delta_3 = \left(\Delta_2 \otimes \mathbf{1}\right)\Delta_2 - \left(\mathbf{1} \otimes \Delta_2\right)\Delta_2\\
&\Delta_4\partial - \partial\Delta_4 = \left(\Delta_3 \otimes \mathbf{1}\right)\Delta_2 + \left(\mathbf{1} \otimes \Delta_3\right)\Delta_2 - \left(\Delta_2 \otimes \mathbf{1} \otimes \mathbf{1}\right)\Delta_3\\
&\qquad\qquad\qquad\left(\mathbf{1} \otimes \Delta_2 \otimes \mathbf{1}\right)\Delta_3 - \left(\mathbf{1} \otimes \mathbf{1} \otimes \Delta_2\right)\Delta_3.
\end{aligned}$$

Note that the first relation written in the form $\left(\partial \otimes \mathbf{1} + \mathbf{1} \otimes \partial\right)\Delta_2 = \Delta_2\partial$ says that ∂ is a coderivation of Δ_2 and is dual to the Leibniz Rule $dm = m\left(d \otimes 1 + 1 \otimes d\right)$ in calculus.

In the discussion that follows, we will use the *Sign Commutation Rule*, which states the following:

Sign Commutation Rule: If an object of degree p passes an object of degree q, it introduces the sign $(-1)^{pq}$.

This thesis presents an A_∞-coalgebra structure on the cellular chains of an *n-gon*. This structure is interesting because the operation Δ_k is non-trivial for all $k < n$ and vanishes for all $k \geq n$.

2 Statement of Main Result

Let P be a counterclockwise oriented polygon with n sides where $n \geq 3$. Label the vertices $v_1, v_2, \ldots v_n$ and the edges $e_1, e_2, \ldots e_n$ as in the following diagram:

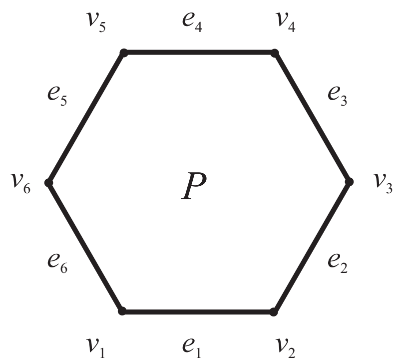

Figure 3: An n-gon for $n = 6$

Next we define one of the vertices to be the *initial vertex* and one of them to be the *terminal vertex*. This specifies a partially ordered set where all vertices along one side of the polygon are ordered in a chain from the initial vertex to the terminal vertex, and all vertices along the other are ordered in a similar chain, but vertices in different chains cannot be compared to each other. For example, suppose we had an 6-gon and we labeled v_1 to be the initial vertex and v_3 to be the terminal vertex. Then we would have that $v_1 < v_2 < v_3$ and $v_1 < v_6 < v_5 < v_4 < v_3$. Because of this, we sometimes call the initial vertex the *minimal vertex* and the terminal vertex the *maximal vertex*.

For our proof, we will start by labeling v_1 as the initial vertex and v_n as the terminal vertex, but later on we will relax this condition when we prove our corollary in section 5. Next, we assign an orientation to each edge based off this ordering, and we say that edges with direction consistent with the counterclockwise orientation of P are positive and those that go against it are negative.

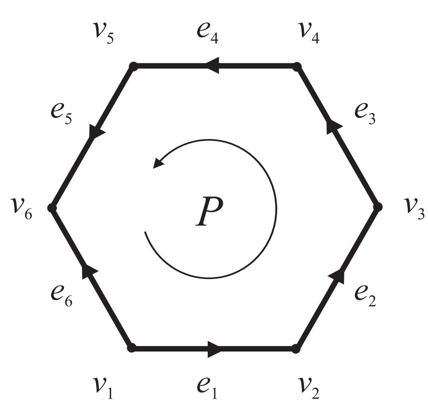

Figure 3: An n-gon for $n = 6$

Then $\partial(v_i) = 0$, $\partial(e_i) = v_{i+1} - v_i$ if $i < n$, $\partial(e_n) = v_n - v_1$, and $\partial(p) = e_1 + e_2 + \cdots + e_{n-1} - e_n$.

The Δ_2 that we use here was first discovered by Kravatz in [2]. A more general exposition of Δ_2 subsequently appeared in [1]. Define the diagonal Δ_2 by:

$$\begin{aligned}
\Delta_2(P) &= v_1 \otimes P + P \otimes v_n + \sum_{0<i_1<i_2<n} e_{i_1} \otimes e_{i_2}, \\
\Delta_2(e_i) &= v_i \otimes e_i + e_i \otimes v_{i+1} \text{ if } i < n, \\
\Delta_2(e_n) &= v_1 \otimes e_n + e_n \otimes v_n, \\
\Delta_2(v_i) &= v_i \otimes v_i,
\end{aligned}$$

and define the k-ary A_∞-coalgebra operations Δ_k where $2 < k$ as:

$$\begin{aligned}
\Delta_k(P) &= \sum_{0<i_1<i_2<\cdots<i_k<n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \\
\Delta_k(\sigma) &= 0 \text{ when } \sigma \neq P.
\end{aligned}$$

Note that by definition, $\Delta_k = 0$ for all $k \geq n$. Recall that $(C_*(P), \partial, \Delta_2, \Delta_3, \cdots)$ is an A_∞-coalgebra if the following relation holds for $k \geq 2$:

$$\sum_{i=0}^{k-1} \left(1^{\otimes i} \otimes \partial \otimes 1^{\otimes k-i-1}\right) \Delta_k - (-1)^p \Delta_k \partial = \sum_{l=1}^{k-2} \sum_{i=0}^{k-l-1} (-1)^{l(k+i+1)} \left(1^{\otimes i} \otimes \Delta_{l+1} \otimes 1^{\otimes k-l-i-1}\right) \Delta_{k-l}. \tag{2}$$

Our main result, which appears in section 4, is:

Theorem 6 *The operations defined above satisfy all A_∞-coalgebra relations on $C_*(P)$, the cellular chains of P. Furthermore, all Δ_k for $k < n$ are non-trivial, and all Δ_k for $k \geq n$ vanish.*

3 Proof that ∂ is a coderivative of Δ_2

The first A_∞-coalgebra relation states that ∂ is a coderivative of Δ_2. Thus, we must verify that

$$\Delta_2\partial - \partial\Delta_2 = 0 \tag{3}$$

for all linear combinations of the cellular chains of P. Verifying the relation for edges and vertices can be done easily as follows:

Proposition 7 *Let v be a vertex. Then* $(\Delta_2\partial - \partial\Delta_2)(v) = 0$

Proof.

$$\begin{aligned}
&\left(\Delta_2\partial - \partial\Delta_2\right)(v)\\
=&\Delta_2\partial(v) - (\partial\otimes\mathbf{1} + \mathbf{1}\otimes\partial)\Delta_2(v)\\
=&\Delta_2(0) - (\partial\otimes\mathbf{1} + \mathbf{1}\otimes\partial)(v\otimes v) = 0
\end{aligned}$$

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Proposition 8 *Let e_i be an edge of P. Then* $(\Delta_2\partial - \partial\Delta_2)(e_i) = 0$

Proof.

Case 1: Suppose $i < n$. Then

$$\begin{aligned}
&(\Delta_2\partial - \partial\Delta_2)(e_i)\\
=&\Delta_2\partial(e_i) - \partial\Delta_2(e_i)\\
=&\Delta_2(v_{i+1} - v_i) - (\partial\otimes\mathbf{1} + \mathbf{1}\otimes\partial)(v_i\otimes e_i + e_i\otimes v_{i+1})\\
=&v_{i+1}\otimes v_{i+1} - v_i\otimes v_i - 0\otimes e_i - v_i\otimes(v_{i+1} - v_i) - (v_{i+1} - v_i)\otimes v_{i+1} - e_i\otimes 0 = 0
\end{aligned}$$

Case 2: Suppose $i = n$. Then

$$\begin{aligned}
&(\Delta_2\partial - \partial\Delta_2)(e_n)\\
=&\Delta_2\partial(e_n) - \partial\Delta_2(e_n)\\
=&\Delta_2(v_1 - v_n) - (\partial\otimes\mathbf{1} + \mathbf{1}\otimes\partial)(v_1\otimes e_n + e_n\otimes v_n)\\
=&v_1\otimes v_1 - v_n\otimes v_n - 0\otimes e_n - v_1\otimes(v_1 - v_n) - (v_1 - v_n)\otimes v_n - e_n\otimes 0 = 0
\end{aligned}$$

■

Finally, we must verify relation (3) when applied to P. In order to make our proof clearer however, we first prove a lemma.

Lemma 9 *Let v_1 and v_n be vertices of P and let $e_1, e_2, \ldots, e_{n-1}$ be edges of P. Then*

$$\partial^{\otimes 2}\left(\sum_{0<i<j<n} (e_i \otimes e_j)\right) = \left(-\sum_{0<i<n} (v_1 \otimes e_i) - \sum_{0<i<n} (e_i \otimes v_n) + \sum_{0<i<n} \Delta_2(e_i)\right)$$

Proof. We apply $\partial^{\otimes 2}$ and simplifying:

$$\begin{aligned}
&\partial^{\otimes 2}\left(\sum_{0<i<j<n} (e_i \otimes e_j)\right) \\
&= \sum_{0<i<j<n} (\partial e_i \otimes e_j - e_i \otimes \partial e_j) \\
&= \sum_{0<i<j<n} (v_{i+1} \otimes e_j - v_i \otimes e_j - e_i \otimes v_{j+1} + e_i \otimes v_j) \\
&= \sum_{0<i<j<n} (v_{i+1} \otimes e_j - v_i \otimes e_j) + \sum_{0<i<j<n} (e_i \otimes v_j - e_i \otimes v_{j+1}) \\
&= \sum_{1<j<n} \sum_{0<i<j} (v_{i+1} \otimes e_j - v_i \otimes e_j) + \sum_{0<i<n-1} \sum_{i<j<n} (e_i \otimes v_j - e_i \otimes v_{j+1}) \\
&= \sum_{1<j<n} (v_j \otimes e_j - v_1 \otimes e_j) + \sum_{0<i<n-1} (e_i \otimes v_{i+1} - e_i \otimes v_n) \\
&= -\sum_{1<j<n} v_1 \otimes e_j + \sum_{1<j<n} v_j \otimes e_j + \sum_{0<i<n-1} e_i \otimes v_{i+1} - \sum_{0<i<n-1} e_i \otimes v_n \\
&= -\sum_{0<j<n} v_1 \otimes e_j + \sum_{0<j<n} v_j \otimes e_j + \sum_{0<i<n} e_i \otimes v_{i+1} - \sum_{0<i<n} e_i \otimes v_n \\
&= -\sum_{0<j<n} v_1 \otimes e_j - \sum_{0<i<n} e_i \otimes v_n + \sum_{0<i<n} e_i \otimes v_{i+1} + \sum_{0<j<n} v_j \otimes e_j \\
&= -\sum_{0<j<n} v_1 \otimes e_j - \sum_{0<i<n} e_i \otimes v_n + \sum_{0<i<n} (v_i \otimes e_i + e_i \otimes v_{i+1}) \\
&= -\sum_{0<j<n} v_1 \otimes e_j - \sum_{0<i<n} e_i \otimes v_n + \sum_{0<i<n} \Delta_2 e_i
\end{aligned}$$

■

Proposition 10 *Relation (3) holds for P*

Proof.

$$\begin{aligned}
&[\Delta_2\partial - \partial\Delta_2](P)\\
=&\Delta_2\partial(P) - \partial\Delta_2(P)\\
=&\Delta_2\partial(P) - \partial\left[v_1 \otimes P + P \otimes v_n + \sum_{0<i_1<i_2<n} (e_{i_1} \otimes e_{i_2})\right]\\
=&\Delta_2\partial(P) - \partial[v_1 \otimes P + P \otimes v_n] - \partial\left[\sum_{0<i_1<i_2<n} (e_{i_1} \otimes e_{i_2})\right]\\
=&\Delta_2\partial(P) - [v_1 \otimes \partial(P) + \partial(P) \otimes v_n] - \partial\left[\sum_{0<i_1<i_2<n} (e_{i_1} \otimes e_{i_2})\right]\\
=&\Delta_2\partial(P) - v_1 \otimes \partial(P) - \partial(P) \otimes v_n - \left[-\sum_{0<i<n} (v_1 \otimes e_i) - \sum_{0<i<n} (e_i \otimes v_n) + \sum_{0<i<n} \Delta_2(e_i)\right]\\
=&\Delta_2(-e_n) - v_1 \otimes \partial(P) - \partial(P) \otimes v_n + \sum_{0<i<n} (v_1 \otimes e_i) + \sum_{0<i<n} (e_i \otimes v_n)\\
=&-\Delta_2(e_n) - v_1 \otimes \partial(P) - \partial(P) \otimes v_n + v_1 \otimes \left(\sum_{0<i<n} e_i\right) + \left(\sum_{0<i<n} e_i\right) \otimes v_n\\
=&-\Delta_2(e_n) - v_1 \otimes \partial(P) - \partial(P) \otimes v_n + v_1 \otimes (\partial(P) + e_n) + (\partial(P) + e_n) \otimes v_n\\
=&-\Delta_2(e_n) + v_1 \otimes e_n + e_n \otimes v_n = 0
\end{aligned}$$

■

4 Proof that Δ_n satisfies all A_∞-coalgebra relations for $n > 2$

Since ∂ is a coderivative of Δ_2, it is only necessary to justify $\Delta_k(P)$ for $k > 2$. Verifying Δ_3 turns out to be a special case and relations with $k > n$ involve vanishing Δ_i's. Consequently we begin by proving cases for $3 < k \leq n$, followed by $n < k$, and conclude with the special case Δ_3.

For the rest of the discussion, we will be verifying the A_∞-coalgebra relations when Δ_k is applied to P (applying Δ_k to other cells always vanishes, and the conclusion is trivial). To prove our result for Δ_k when $k > 3$, we must show that:

$$
\begin{aligned}
&[\partial\Delta_k - (-1)^{k-2}\Delta_k\partial](P) \\
&\qquad = (\Delta_{k-1}\otimes\mathbf{1} + \mathbf{1}\otimes\Delta_{k-1})\Delta_2 + (\Delta_{k-2}\otimes\mathbf{1}\otimes\mathbf{1} + \mathbf{1}\otimes\Delta_{k-2}\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{1}\otimes\Delta_{k-2})\Delta_3 \\
&\qquad + \cdots + (\Delta_2\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} + \cdots + \mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\Delta_2)\Delta_{k-1}](P).
\end{aligned}
$$

Each term on the right side of this equation has the sign $(-1)^{l(k+i+1)}$, where l is one less than the degree of the second operation applied, and i represents the position where the second operation is applied (starting at position 0).

Proposition 11 *Any sequence of compositions of Δ_j followed by Δ_j where $i, j > 2$ applied to P vanishes. Additionally, $\Delta_k\partial(P) = 0$.*

Proof. Since Δ_i only act on P, and since P is not a factor of $\Delta_j(P)$, any composition of Δ_j followed by Δ_i will vanish. By the same reasoning, $\Delta_k\partial(P)$ vanishes as well since $\partial(P)$ only involves edges. ■

Therefore, the proof reduces to verifying that:

$$
\begin{aligned}
&-(-1)^k\partial\Delta_k(P) = \\
&\qquad [(\Delta_{k-1}\otimes\mathbf{1} + \mathbf{1}\otimes\Delta_{k-1})\Delta_2 + (\Delta_2\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} + \cdots + \mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\Delta_2)\Delta_{k-1}](P),
\end{aligned}
\tag{4}
$$

where the signs on the right are determined by $(-1)^{l(k+i+1)}$. To simplify notation, we will use the following:

- Let $\partial\Delta_k(P)$ denote: $(\partial\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} + \cdots + \mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\partial)\Delta_k(P)$
- Let $\Delta_{k-1}\Delta_2(P)$ denote: $(\Delta_{k-1}\otimes\mathbf{1} + \mathbf{1}\otimes\Delta_{k-1})\Delta_2(P)$
- Let $\Delta_2\Delta_{k-1}(P)$ denote: $(\Delta_2\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} + \cdots + \mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\Delta_2)\Delta_{k-1}(P)$

So in other words, we must show that:

$$
[(-1)^{k+1}\partial\Delta_k + (-1)^{l(k+i+1)+1}\Delta_{k-1}\Delta_2 + (-1)^{l(k+i+1)+1}\Delta_2\Delta_{k-1}](P) = 0, \tag{5}
$$

where all terms in relation (4) have been moved to the left-hand side of the equation. Before we begin, we note the restrictions on the types of terms to be considered.

Lemma 12 *All non-vanishing terms contain exactly one v_j factor. Furthermore, if $e_i \otimes v_j$ or $v_j \otimes e_k$ appears as part of a term, then $i < j$ and $j \leq k$, respectively.*

Proof.

Case 1: Consider $\partial\Delta_k(P)$. Note that $\Delta_k(P)$ will either vanish if $k \geq n$, or it will produce terms of the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ where $0 < i_1 < i_2 < \cdots < i_k < n$. The effect of applying ∂ to any e_i is to create two new terms, where e_i is replaced with v_i or v_{i+1}. Since the subscript i is greater than the subscript on the term to its left and less than the subscript of the term to its right, either choice gives the desired result.

Case 2: Consider $\Delta_{k-1}\Delta_2(P)$. Note that since any non-vanishing Δ_{k-1} only acts nontrivial on P, the only terms produced by Δ_2 that we need to consider are $v_1 \otimes P$ and $P \otimes v_n$. When Δ_{k-1} acts on the P, it produces a term of the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{k-1}}$ where $0 < i_1 < i_2 < \cdots < i_{k-1} < n$. Since $0 < i < n$ for all i, all terms either begin with v_1 or end with v_n and have the desired form.

Case 3: Consider $\Delta_2\Delta_{k-1}(P)$. Notice that non-vanishing $\Delta_{k-1}(P)$ produces terms of the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{k-1}}$ where $0 < i_1 < i_2 < \cdots < i_{k-1} < n$. When Δ_2 is applied, it acts on a factor e_i, which will have the effect of either inserting v_i to the left of the e_i, or of inserting v_{i+1} to the right of the e_i. Since the subscript i is bigger than the subscript of the term to its left, inserting a v_i to the left of the e_i produces a term of the desired form. Additionally, since the subscript i is less than the subscript on the term to its right, inserting a v_{i+1} to the right of the e_i produces a term of the desired form.

■

With this information, we can partition all terms into several disjoint sets based on the position of the v_i. We say that v_i is *left adjacent* if e_{i-1} is immediately to its left; v_i is *right adjacent* if e_i is immediately to its right. Then we may partition the terms as follows

Definition 13 *A term in which v_i is both left and right adjacent is doubly attached.*

Definition 14 *A term in which v_i is left adjacent but not right adjacent or vice versa is singly attached.*

Definition 15 *A term in which v_i is neither left adjacent nor right adjacent is unattached.*

These definitions partition all terms into pairwise disjoint sets. Each term satisfies exactly one of the definitions. Additionally, we can sub-classify these sets using the following definition:

Definition 16 *A term of the form $v_1 \otimes e_j \otimes \cdots$ is called an extreme minimal term. A term of the form $\cdots \otimes e_j \otimes v_n$, is called an extreme maximal term. An extreme term either starts with v_1 or ends with v_n.*

Note that doubly attached terms cannot be extreme. Hence, there are five classes of terms.

Example 17 *For $n = 6$, we have:*

- $v_1 \otimes e_1 \otimes e_3$ *is minimally extreme and right attached*
- $e_1 \otimes e_5 \otimes v_6$ *is maximally extreme and left attached*
- $v_1 \otimes e_2 \otimes e_3$ *is minimally extreme and unattached*
- $e_1 \otimes e_4 \otimes v_6$ *is maximally extreme and unattached*
- $v_2 \otimes e_2 \otimes e_5$ *is right attached (but not minimally extreme)*
- $e_1 \otimes e_4 \otimes v_5$ *is left attached (but not maximally extreme)*
- $v_2 \otimes e_3 \otimes e_5$ *is unattached*
- $e_1 \otimes e_3 \otimes v_5$ *is unattached*
- $e_1 \otimes v_2 \otimes e_3$ *is left attached*

- $e_1 \otimes v_4 \otimes e_4$ *is right attached*

- $e_2 \otimes v_3 \otimes e_3$ *is doubly attached*

- $e_1 \otimes v_3 \otimes e_4$ *is unattached*

The proof now reduces to showing that each class cancels itself out in the relation:

$$[-(-1)^k \partial \Delta_k + (-1)^{l(k+i+1)+1} \Delta_{k-1} \Delta_2 + (-1)^{l(k+i+1)+1} \Delta_2 \Delta_{k-1}](P) = 0.$$

But before we can do this however, we need some lemmas.

Lemma 18 *Suppose a term is formed by applying Δ_k to P (for $2 < k < n$), followed by applying ∂ in such a way so that v_i appears in the i^{th} position. Then that term appears at most two times and only in the following ways:*

- ***Way 1:*** *The term will be generated once if and only if it is unattached on the left and is not minimally extreme (note that terms beginning with any v_i, where $i \neq 1$, are unattached on the left). This term will have the sign $(-1)^{i+k+1}$.*

- ***Way 2:*** *The term will be generated once if and only if it is unattached on the right and is not maximally extreme (note that terms ending with any v_i, where $i \neq n$, are unattached on the right). This term will have the sign $(-1)^{i+k}$.*

Proof.

Case 1: Consider a term that contains $e_{i_1} \otimes v_{i_2} \otimes e_{i_3}$ where $0 < i_1 < i_2 \leq i_3 < n$. Since $\Delta_k(P)$ cannot generate v_{i_2}, it was produced by ∂. Since v_{i_2} appears between e_{i_1} and e_{i_3}, we know that ∂ was applied to a term of the form $e_{i_1} \otimes e_x \otimes e_{i_3}$ with $0 < i_1 < x < i_3 < n$. Furthermore, there are only two possible edges e_x that ∂ could have acted upon to generate v_{i_2}, namely when $x = i_2 - 1$ or $x = i_2$. Therefore, since there are only two possible values for x, there are at most two ways to generate any given term. For example, the term $e_1 \otimes v_3 \otimes e_4$ can only be generated from $e_1 \otimes \partial e_2 \otimes e_4$ or $e_1 \otimes \partial e_3 \otimes e_4$.

Suppose that $x = i_2 - 1$. Since $i_1 < x$, we have $i_1 < i_2 - 1$ forcing the original term to be unattached on the left. Therefore, only terms that are unattached on the left can appear this way. To see that all left unattached terms can be generated, simply note that the subscript of any left unattached term will satisfy the inequality $i_1 < i_2 - 1$ by definition. Then $\Delta_k(P)$ produces the term $e_{i_1} \otimes e_{i_2-1} \otimes e_{i_3}$ where $0 < i_1 < i_2 \leq i_3 < n$, and produces the desired term after ∂ is applied to e_{i_2-1}. Now applying ∂ this way produces a positive v_{i_2} in the same position in which ∂ was applied, and so if v_i appears in the i^{th} position, then a sign of $(-1)^i$ is introduced from ∂ passing the e_j factors. If we combine this with the sign given in the formula, the term's final sign is $(-1)^{i+k+1}$.

Similarly, suppose that $x = i_2$. Since $x < i_3$, we have $i_2 < i_3$ which forces the original term to be unattached on the right. Therefore, only terms that are unattached on the right can appear this way. To see that all right unattached terms can be generated, simply note that the subscript of any right unattached term will satisfy the inequality $i_1 < i_2$ by definition. Thus, $\Delta_k(P)$ produces the term $e_{i_1} \otimes e_{i_2} \otimes e_{i_3}$ where $0 < i_1 < i_2 \leq i_3 < n$, and produces the desired form after applying ∂. Now applying ∂ produces a negative v_{i_2} in the same position in which ∂ was applied. Consequently, if v_i appears in the i^{th} position, then a sign of $(-1)^i$ is introduced from ∂ passing the e_j factors. If we combine this with the sign given in the formula and the negative sign introduced by the v_{i_2}, we have that the term's final sign is $-(-1)^{i+k+1} = (-1)^{i+k}$.

Case 2: Consider a term of the form $v_{i_2} \otimes e_{i_3} \otimes \cdots$ where $0 < i_2 \leq i_3 < n$. By an argument similar to the one above, this term can only appear if ∂ is applied to a term of the form $e_x \otimes e_{i_3} \otimes \cdots$, where $x = i_2 - 1$ or $x = i_2$. For example, the term $v_3 \otimes e_4 \otimes e_5$ can only be generated from $\partial e_2 \otimes e_4 \otimes e_5$ or $\partial e_3 \otimes e_4 \otimes e_5$. By the same reasoning as in Case 1, it follows that if $x = i_2$, then all terms generated must be right unattached, any given right unattached term can be formed, and the term will have the sign $(-1)^{i+k}$. Similarly, if $x = i_2 - 1$ all terms introduced are left unattached, however, we cannot immediately conclude that all possible left unattached terms will be formed. This is because $x > 0$, and since $x = i_2 - 1$, we have $i_2 > 1$. Hence only non-minimally extreme left unattached terms can be formed, and an analysis similar to that in Case 1 shows that the terms produced have the sign $(-1)^{i+k+1}$.

Case 3: Consider a term of the form $\cdots \otimes e_{i_1} \otimes v_{i_2}$ where $0 < i_1 < i_2 < n$. By an argument similar to the one above, this term can only appear if ∂ is applied to a term of the form $\cdots \otimes e_{i_1} \otimes e_x$, where $x = i_2 - 1$ or $x = i_2$. For example, the term $e_1 \otimes e_2 \otimes v_4$ can only be generated from $e_1 \otimes e_2 \otimes \partial e_3$ or $e_1 \otimes e_2 \otimes \partial e_4$. By the same reasoning as in Case 1, it follows that if $x = i_2 - 1$, then all terms generated must be left unattached, any given left unattached term can be formed, and the the term will have the sign $(-1)^{i+k+1}$. Similarly, if $x = i_2$ all terms produced are right unattached, however we cannot immediately conclude that all possible right unattached terms will be formed. This is because $x < n$, and since $x = i_2$, we have $i_2 < n$. Hence, only non-maximally extreme right unattached terms can be formed, and an analysis similar to that in Case 1 shows that the terms produced have the sign $(-1)^{i+k}$. ■

Lemma 19 *Let $3 < k \leq n$. Suppose a term is produced by applying Δ_{k-1} to P followed by applying Δ_2 in such a way so that a v_i appears in the i^{th} position. Then that term appears at most two times and only in the following ways:*

- ***Way 1:*** *The term will be generated once if and only if it is attached on the left. This term will have the sign $(-1)^{i+k+1}$.*
- ***Way 2:*** *The term will be generated once if and only if it is attached on the right. This term will have the sign $(-1)^{i+k}$.*

Proof.

Case 1: Consider a term that contains $e_{i_1} \otimes v_{i_2} \otimes e_{i_3}$ where $0 < i_1 < i_2 \leq i_3 < n$. Since $\Delta_{k-1}(P)$ cannot produce v_{i_2}, it is produced by Δ_2, and because Δ_2 inserts a v next to the e to which it was applied, Δ_2 was applied to either e_{i_1} or e_{i_3}. Additionally, because of how Δ_2 is defined, the only way $\Delta_2(e_{i_1})$ can generate v_{i_2} is if $i_1 = i_2 - 1$, and the only way $\Delta_2(e_{i_3})$ can generate v_{i_2} is if $i_2 = i_3$. Therefore, there are at most two ways to create the desired term. For example, the term $e_1 \otimes e_2 \otimes v_3 \otimes e_3 \otimes e_5$ can only be generated from $e_1 \otimes \Delta_2 e_2 \otimes e_3 \otimes e_5$ or $e_1 \otimes e_2 \otimes \Delta_2 e_3 \otimes e_5$.

Let us suppose that Δ_2 was applied to e_{i_1} and that $i_1 = i_2 - 1$. Then by definition the resulting term will be left attached. Furthermore, any given left attached term of the form $e_{i_2-1} \otimes v_{i_2} \otimes e_{i_3}$ where $0 < i_2 - 1 \leq i_3 < n$ can be created by simply applying Δ_2 to the term $e_{i_2-1} \otimes e_{i_3}$ generated by $\Delta_k(P)$. Now in order to create a term with v_{i_2} in the i^{th} position, we must apply Δ_2 to the $(i-1)^{st}$ position. Since Δ_2 has even degree, no additional sign is introduced by the sign commutation rule and the term has the sign from the formula, namely $(-1)^{l((i-1)+k+1)+1} = (-1)^{i+k+1}$ since $l = 1$.

Now suppose that Δ_2 was applied to e_{i_3} and that $i_2 = i_3$. Then by definition the resulting term will be right attached, and additionally we can see that any given right attached term of the form $e_{i_1} \otimes v_{i_2} \otimes e_{i_2}$ where $0 < i_2 < n$ can be created by simply applying Δ_2 to the term $e_{i_1} \otimes e_{i_2}$ generated by $\Delta_k(P)$. Now in order to create a term with v_{i_2} in the i^{th} position, we must apply Δ_2 to the i^{th} position. Since Δ_2 has even degree, no additional sign is introduced by the sign commutation rule and the term has the sign from the formula, namely $(-1)^{l(i+k+1)+1} = (-1)^{i+k}$ since $l = 1$.

Case 2: Consider a term of the form $v_{i_2} \otimes e_{i_3} \otimes \cdots$ where $0 < i_2 < i_3 < n$. By an argument similar to the one above, this term can only be generated if Δ_2 is applied to e_{i_3} and if $i_2 = i_3$. For example, the term $v_3 \otimes e_3 \otimes e_5$ can only be generated from $\Delta_2 e_3 \otimes e_5$. Using the same reasoning as in Case 1, all terms generated by this method will be right attached, and any given right attached term can be generated. The sign of the term, which follows in the same manner as above, is $(-1)^{i+k}$.

Case 3: Consider a term of the form $\cdots \otimes ei_1 \otimes v_{i_2}$ where $0 < i_1 < i_2 < n$. By an argument similar to the one above, this term can only be generated if Δ_2 is applied to e_{i_1} and if $i_1 = i_2 - 1$. For example, the term $e_1 \otimes e_2 \otimes v_3$ can only be generated from $e_1 \otimes \Delta_2 e_2$. Using the same reasoning as in Case 1, all terms generated by this method will be left attached, and any given left attached term can be generated. The sign of the term, which follows in the same manner as above, is $(-1)^{i+k+1}$. ■

Proposition 20 *Let $2 < k < n$ Suppose a term is produced by applying Δ_k to P, followed by applying ∂ in such a way so that a v_i appears in the i^{th} position. Then:*

- *No doubly attached terms are generated.*
- *All unattached terms that are produced cancel.*
- *All extreme terms generated are unattached.*
- *All maximally extreme terms are generated and have positive sign.*
- *All minimally extreme terms are generated and have the sign $(-1)^k$.*

- *All left attached terms are generated and each has the sign* $(-1)^{i+k}$.

- *All right attached terms are generated and each has the sign* $(-1)^{i+k+1}$.

Proof. First, note that both ways of generating a term require it to be unattached on at least one side by Lemma 18. Therefore no doubly attached terms are generated. Additionally, any term that is unattached on both sides with be created in both ways, and since the two ways generate terms with opposite signs, all unattached terms cancel. If a term is attached on exactly one side, it was produced exactly once with sign given by Lemma 18 depending on which side it was attached. Finally, let us consider extreme terms. By the conclusion of Lemma 18, we see that any minimally extreme term attached on the right along with any maximally extreme term attached on the left cannot be generated. If an extreme term is unattached however, it can be generated in exactly one way with sign given by Lemma 18 (maximally extreme terms are generated using the first method, and minimally extreme are generated using the second). Since minimally extreme terms produce a v_i in the zeroth position, we have $i = 0$ which gives the term a sign of $(-1)^k$. On the other hand, maximally extreme terms produce a v_i in the $k-1$ position, so that $i = k-1$, which gives the term the positive sign of $(-1)^{2k}$. ∎

Proposition 21 *Let* $3 < k \leq n$. *Suppose a term is formed by applying* Δ_2 *to* P, *followed by applying* Δ_{k-1}. *Then:*

- *All terms generated are extreme.*

- *All minimally extreme terms (both attached and unattached) are generated and have the sign* $(-1)^{k+1}$.

- *All maximally extreme terms (both attached and unattached) are generated and have negative the sign.*

Proof. Since Δ_{k-1} only acts non-trivially on P, the only terms produced by Δ_2 that do not immediately vanish are $v_1 \otimes P + P \otimes v_n$. Additionally, note that since $k > 3$, Δ_{k-1} produces no primitive terms when it is applied by the definition of Δ_k for $k > 2$.

When Δ_{k-1} is applied to the first term, it generates terms of the form $v_1 \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots$, which are all minimally extreme. Since Δ_{k-1} must be applied to the second position, it passes v_1 which has an even degree, so no additional sign is added by the sign commutation rule and the term has the final sign $(-1)^{l(k+i+1)+1} = (-1)^{lk+1}$. Furthermore, since $l = k-2$, we see that l and k have the same parity, and the sign is simply $(-1)^{k+1}$.

When Δ_{k-1} is applied to the second term, it generates terms of the form $\cdots \otimes e_{i_{k-3}} \otimes e_{i_{k-2}} \otimes v_n$, which are maximally extreme. Since Δ_{k-1} must be applied to the first position, no additional sign is introduced by the sign commutation rule and the term has the final sign $(-1)^{l(k+i+1)+1} = (-1)^{l(k+1)+1}$. Furthermore, since l and k have the same parity as noted before, the expression $l(n+1)$ is even, and all terms have negative sign. ■

Proposition 22 *Let $3 < k \leq n$. Suppose a term is formed by applying Δ_{k-1} to P, followed by applying Δ_2 in such a way so that v_i appears in the i^{th} position. Then:*

- *All doubly attached terms that are generated cancel.*
- *No unattached terms are generated.*
- *All extreme terms generated are singly attached.*
- *All maximally extreme terms are generated and have positive sign.*
- *All minimally extreme terms are generated and have the sign $(-1)^k$.*
- *All left attached terms have the sign $(-1)^{i+k+1}$.*
- *All right attached terms have the sign $(-1)^{i+k}$.*

Proof. First, note that both ways of generating a term require it to be attached on at least one side by Lemma 19. Therefore no unattached terms can be generated. Additionally, any term that is attached on both sides is created both ways, and since the two ways generate terms with opposite signs, all doubly attached terms cancel. If a term is attached on exactly one side, it was produced exactly once with sign given by Lemma 19 depending on which side it was attached. Finally, let us consider extreme terms. By the conclusion of Lemma 19, any extreme term cannot be generated unless it is attached, which means that minimally extreme terms must be attached on the right (and are therefore produced by the second method described in the statement of the lemma), and any maximally extreme terms will be attached on the left (and are therefore produced by the first method described in the statement of the lemma). Since minimally extreme terms will result in v_i in the zeroth position, we have that $i = 0$ which gives the term the sign of $(-1)^k$. On the other hand, maximally extreme terms will result in v_i in the $(k-1)$ position, so that $i = k-1$, which gives the term the positive sign of $(-1)^{2k}$. ■

Now suppose $3 < k < n$. Then all three propositions apply and we see that when we examine all terms generated in relation (5), the singly attached terms in $\partial\Delta_k(P)$ and $\Delta_2\Delta_{k-1}(P)$ cancel, the extreme unattached terms in $\partial\Delta_k(P)$ and $\Delta_{k-1}\Delta_2(P)$ cancel, and the extreme singly attached terms in $\Delta_{k-1}\Delta_2(P)$ and $\Delta_2\Delta_{k-1}(P)$ cancel. Therefore, nothing remains and the relation is satisfied.

Now suppose that $n \leq k$. Note that if $n < k$, both Δ_k and Δ_{k-1} will vanish and the relation will be trivially satisfied, so we really only need to verify the relation for $k = n$. In this case, Δ_k vanishes and the relation reduces to

$$[(-1)^{l(n+i+1)+1}\Delta_{n-1}\Delta_2 + (-1)^{l(n+i+1)+1}\Delta_2\Delta_{n-1}](P) = 0$$

Notice that Proposition 21 and Proposition 22 still apply since $k \leq n$, and as before, the extreme singly attached terms in $\Delta_{k-1}\Delta_2(P)$ and $\Delta_2\Delta_{k-1}(P)$ cancel, leaving only the singly attached terms in $\Delta_2\Delta_{k-1}(P)$, and the extreme unattached terms in $\Delta_{k-}\Delta_2(P)$. We will now show that no singly attached terms or extreme unattached terms can actually be generated when $k = n$, which will complete the proof.

First of all, notice that each set of operations will produce a tensor product with n factors, and that exactly one of those factors will be a v_i. This means that $n-1$ of the factors must be e_i where the index of each e_i must be greater than the one preceding it, and where $0 < i < n$. Since there are $n-1$ such factors, we have that each of $e_1, e_2, ...e_{n-1}$ must be present in any term generated by either sequence of operations. Now if the term is extreme, the v_i will be located either before the e_1, or after the e_{n-1}. In either case, the term will be singly attached which means extreme unattached terms cannot be generated. Similarly, we have that if the v_i appears somewhere in the middle of the term, then there will be an

e_i on the left and an e_{i+1} on the right, forcing the term to be doubly attached. Therefore, no singly attached terms will be generated either. This means that all the terms which normally would have needed the $\partial\Delta_k(P)$ to cancel were never actually created in the first place, allowing the relation to still be satisfied for $k = n$.

Now we have proven the theorem for all Δ_k where $k > 3$. To prove it for $k = 3$, we must show that:

$$(\partial\Delta_3 + \Delta_2\Delta_2)(P) = 0, \text{ i.e.,}$$

$$[(\partial \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \partial \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \partial)\Delta_3 + (-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\Delta_2](P) = 0$$

when we account for signs.

Proposition 23 *The terms generated by $(\partial\Delta_3 + \Delta_2\Delta_2)(P)$ can be divided into parts that independently satisfy the conclusions of Propositions 20, 21, and 22.*

Proof. Note that Proposition 20 in the proof above still applies to $\partial\Delta_3(P)$ as before since it never used a restriction that $k \geq 3$. Therefore, we must show that $(\Delta_2 \otimes \mathbf{1} - \mathbf{1} \otimes \Delta_2)\Delta_2$ can be divided into parts that separately satisfy the conclusions of Proposition 21 and Proposition 22. Now

$$\begin{aligned}
&(-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\Delta_2(P) \\
&\qquad =(-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\left(v_1 \otimes P + P \otimes v_n + \sum_{i_1,i_2}(e_{i_1} \otimes e_{i_2})\right) \\
&\qquad =(-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\left(v_1 \otimes P + P \otimes v_n\right) + (-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\left(\sum_{i_1,i_2}(e_{i_1} \otimes e_{i_2})\right).
\end{aligned}$$

Notice that the applications $(-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\sum_{i_1,i_2} e_{i_1} \otimes e_{i_2}$ produce terms of the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{k-1}}$ where $0 < i_1 < i_2 < \cdots < i_{k-1} < n$, then applies Δ_2 to one of them. This is exactly the same set up as in Proposition 22, and since the proof of Proposition 22 only used the condition that $k \geq 3$ in order to ensure there we no primitive terms, we see that the proof applies to $(\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2) \sum_{i_1,i_2} e_{i_1} \otimes e_{i_2}$ and that we obtain the same types of terms as given in Proposition 22.

Therefore, we must show that what remains behaves like the general case of $\Delta_{k-1}\Delta_2$, that is, produces all possible extreme terms.

$$
\begin{aligned}
&(-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)(v_1 \otimes P + P \otimes v_n) \\
&\quad = -v_1 \otimes v_1 \otimes P + P \otimes v_n \otimes v_n - \Delta_2(P) \otimes v_n + v_1 \otimes \Delta_2(P) \\
&\quad = -v_1 \otimes v_1 \otimes P + P \otimes v_n \otimes v_n - \left(v_1 \otimes P + P \otimes v_n + \sum_{i_1,i_2} e_{i_1} \otimes e_{i_2} \right) \otimes v_n \\
&\qquad + v_1 \otimes \left(v_1 \otimes P + P \otimes v_n + \sum_{i_1,i_2} e_{i_1} \otimes e_{i_2} \right) \\
&\quad = -\left(\sum_{i_1,i_2} e_{i_1} \otimes e_{i_2} \right) \otimes v_n + v_1 \otimes \left(\sum_{i_1,i_2} e_{i_1} \otimes e_{i_2} \right)
\end{aligned}
$$

which generates every possible extreme term and only extreme terms, all with the correct sign for $k = 3$. Therefore, $(\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)(v_1 \otimes P + P \otimes v_{n+1})$ meets the hypothesis of Proposition 21 ■

Therefore, since the terms generated by $(\partial\Delta_3 + \Delta_2\Delta_2)(P)$ can be divided into parts that individually satisfy the conclusions of Proposition 20, Proposition 21, and Proposition 22, we see that together they cancel each other out as before and so Δ_3 also satisfies the relations.

Theorem 24 *The operations $\{\Delta_n\}_{n\geq 2}$ defined in the introduction satisfy all A_∞-coalgebra relations on cellular chains of P.*

5 Generalization of Result

Up until this point, we have been working with an n-gon where v_1 is the initial vertex and v_n is the terminal vertex. It is also possible to extend the result to the case where the initial and terminal vertices are non-adjacent. For this result, we will choose some vertex to be the initial vertex, call it v_1, and label all remaining vertices $v_2, v_3, \ldots, v_n$ in a counterclockwise manner. Suppose v_t is then picked to be the terminal vertex where $t \neq 1$ ($t = n$ was our choice above). Then we have that $\partial(P) = e_1 + e_2 + \cdots + e_{t-1} - e_t - e_{t+1} - \cdots - e_n$. Define the generalized Δ_2' to be the same as in the introduction, except that:

$$\Delta_2'(P) = v_1 \otimes P + P \otimes v_t + \sum_{0<i_1<i_2<t} (e_{i_1} \otimes e_{i_2}) - \sum_{n \geq i_1 > i_2 \geq t} (e_{i_1} \otimes e_{i_2})$$

Additionally, define the generalized k-ary A_∞-coalgebra operations Δ_k' where $k > 2$, as:

$$\Delta_k'(P) = \sum_{0<i_1<i_2<\cdots<i_k<t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} - \sum_{n \geq i_1 > i_2 > \cdots > i_k \geq t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \text{ and}$$

$$\Delta_k'(\sigma) = 0 \text{ when } \sigma \neq P.$$

Note that by definition, $\Delta_k' = 0$ for all $k \geq max\{t, n-t+2\}$. Now all we must do is show that the operation $\{\Delta_n'\}_{n \geq 2}$ can be extended to this general setting.

Corollary 25 *The operation $\{\Delta_n'\}_{n \geq 2}$ defined above satisfies all A_∞-coalgebra relations on cellular chains of P where the initial vertex is v_1 and the terminal vertex is v_t where $1 < t \leq n$. Furthermore, all Δ_k for $k < n$ are non-trivial, and all Δ_k for $k \geq max\{t, n-t+2\}$*

Proof. Draw an additional edge from v_1 to v_t and denote it by e_0. Define P_1 to be the polygon with vertices $v_1, v_2, \ldots, v_t$ oriented counterclockwise and let P_2 to be the polygon with vertices $v_t, v_{t+1}, \ldots, v_n, v_1$ oriented counterclockwise.

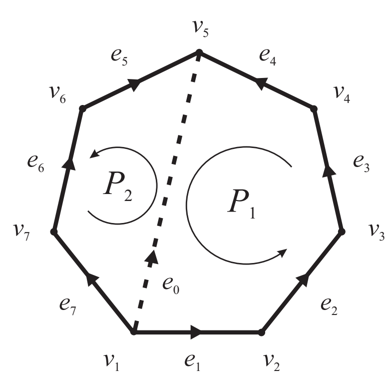


Figure 4: A 7-gon with $v_t = v_5$

Then by the way edges are directed with respect to the orientation, we have $\partial(P_1) =$

$e_1 + e_2 + \cdots + e_{t-1} - e_0$ and $\partial(P_2) = -e_t - e_{t+1} - \cdots - e_n + e_0$, and we note that

$$\begin{aligned}
&\partial(P_1) + \partial(P_2) = (e_1 + e_2 + \cdots + e_{t-1} - e_0) + (-e_t - e_{t+1} - \cdots - e_n + e_0) \\
&= e_1 + e_2 + \cdots + e_{k-1} - e_k - e_{k+1} - \cdots - e_n = \partial(P)
\end{aligned}$$

Furthermore, we define v_1 to be the initial vertex in both P_1 and P_2, and v_t likewise to be the terminal vertex in both P_1 and P_2. Then both P_1 and P_2 satisfy the hypothesis of Theorem 6, and the k-ary operations defined on them define A_∞-coalgebra structures. Now we note the following:

$$\begin{aligned}
&\Delta_2(P_1) + \Delta_2(P_2) \\
&= v_1 \otimes P_1 + P_1 \otimes v_t + \sum_{0<i_1<i_2<t} e_{i_1} \otimes e_{i_2} + v_1 \otimes P_2 + P_2 \otimes v_t - \sum_{n \geq i_1 > i_2 \geq t} (e_{i_1} \otimes e_{i_2}) \\
&= v_1 \otimes (P_1 + P_2) + (P_1 + P_2) \otimes v_t + \sum_{0<i_1<i_2<t} e_{i_1} \otimes e_{i_2} - \sum_{n \geq i_1 > i_2 \geq t} e_{i_1} \otimes e_{i_2} \\
&= v_1 \otimes (P) + (P) \otimes v_t + \sum_{0<i_1<i_2<t} e_{i_1} \otimes e_{i_2} - \sum_{n \geq i_1 > i_2 \geq t} e_{i_1} \otimes e_{i_2} = \Delta_2'(P)
\end{aligned}$$

Additionally, for $k > 2$ we have

$$\begin{aligned}
&\Delta_k(P_1) + \Delta_k(P_2) \\
&= \sum_{0<i_1<i_2<\cdots<i_k<t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} - \sum_{n \geq i_1 > i_2 > \cdots > i_k \geq t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} = \Delta_k'(P)
\end{aligned}$$

Therefore, for all $k \geq 2$, we have that $\Delta_k'(P_1 + P_2) = \Delta_k'(P_1) + \Delta_k(P_2)$

All that remains is to verify that the following relation holds on P for all $k \geq 2$:

$$\sum_{i=0}^{k-1} \left(1^{\otimes i} \otimes \partial \otimes 1^{\otimes k-i-1}\right) \Delta_k' - (-1)^p \, \Delta_k' \partial = \sum_{l=1}^{k-2} \sum_{i=0}^{k-l-1} (-1)^{l(k+i+1)} \left(1^{\otimes i} \otimes \Delta_{l+1}' \otimes 1^{\otimes k-l-i-1}\right) \Delta_{k-l}'. \tag{6}$$

This can be done in the following manner:

$$\sum_{i=0}^{k-1} \left(1^{\otimes i} \otimes \partial \otimes 1^{\otimes k-i-1}\right) \Delta'_k - (-1)^p \, \Delta'_k \partial(P)$$

$$= \sum_{i=0}^{k-1} \left(1^{\otimes i} \otimes \partial \otimes 1^{\otimes k-i-1}\right) \Delta'_k - (-1)^p \, \Delta'_k \partial(P_1 + P_2)$$

$$= \sum_{i=0}^{k-1} \left(1^{\otimes i} \otimes \partial \otimes 1^{\otimes k-i-1}\right) \Delta_k - (-1)^p \, \Delta_k \partial(P_1) + \sum_{i=0}^{k-1} \left(1^{\otimes i} \otimes \partial \otimes 1^{\otimes k-i-1}\right) \Delta_k - (-1)^p \, \Delta_k \partial(P_2)$$

$$= \sum_{l=1}^{k-2} \sum_{i=0}^{k-l-1} (-1)^{l(k+i+1)} \left(1^{\otimes i} \otimes \Delta_{l+1} \otimes 1^{\otimes k-l-i-1}\right) \Delta_{k-l}(P_1)$$

$$+ \sum_{l=1}^{k-2} \sum_{i=0}^{k-l-1} (-1)^{l(k+i+1)} \left(1^{\otimes i} \otimes \Delta_{l+1} \otimes 1^{\otimes k-l-i-1}\right) \Delta_{k-l}(P_2)$$

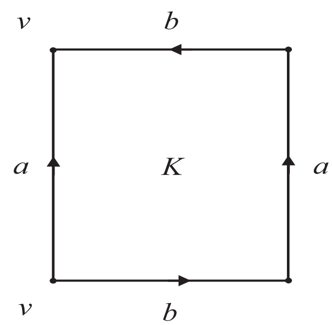
$$= \sum_{l=1}^{k-2} \sum_{i=0}^{k-l-1} (-1)^{l(k+i+1)} \left(1^{\otimes i} \otimes \Delta'_{l+1} \otimes 1^{\otimes k-l-i-1}\right) \Delta'_{k-l}(P_1 + P_2)$$

$$= \sum_{l=1}^{k-2} \sum_{i=0}^{k-l-1} (-1)^{l(k+i+1)} \left(1^{\otimes i} \otimes \Delta'_{l+1} \otimes 1^{\otimes k-l-i-1}\right) \Delta'_{k-l}(P)$$

Therefore, the operation $\{\Delta'_n\}_{n\geq 2}$ defined on P above satisfy all A_∞-coalgebra relations on cellular chains of P. ■

6 Application: A non-trivial A_∞-coalgebra structure on the homology of a Klein bottle.

While we have been defining our A_∞-coalgebra structure on an n-gon with n distinct edges, the same structure still holds even if edges in the n-gon are identified. This, for example, suggests a higher-order coalgebra structure on the Klein bottle as follows.



Using $\mathbb{Z}_2$-coefficients, $\partial K = \partial a = \partial b = \partial v = 0$ so that $H_n(K) = \begin{cases} \mathbb{Z}_2, & n = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & n = 1 \\ \mathbb{Z}_2, & n = 0. \end{cases}$

In homology we have

$$\Delta_2 K = K \otimes v + v \otimes K + a \otimes b + b \otimes b + b \otimes a$$

$$\Delta_3 K = b \otimes a \otimes b.$$

Since the higher-order coalgebra structure on the torus is degenerate, this gives us a way to distinguish between a torus and a Klein bottle using their higher-order structures.

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References

[1] R. Gonzalez-Diaz, J. Lamar, R. Umble. Computing Cup Product in $\mathbb{Z}_2$-cohomology of 3D polyhedral complex. Found. Compact. Math. 14, Springer 2014

[2] D. Kravatz. Diagonal Approximations on an n-gon and the cohomology Ring of Closed Compact Orientable Surfaces. Senior Thesis. Millersville University Department of Mathematics, 2008.
http://sites.millersville.edu/rumble/StudentProjects/Kravatz/finaldraft.pdf

[3] W.S. Massey, Algebraic Topology An Introduction. Harcourt, Brace, and World, Inc., 1967.

[4] S. Saneblidze and R. Umble, Diagonals on the Permutahedra, Multiplihedra and Associahedra, "J. Homology, Homotopy and Appl., 6 (1) (2004), 363-411.

[5] J. D. Stasheff, Homotopy Associativity of H-spaces I, II, Trans. A. M. S. 108 (1963), 275-312.