Periodic Orbits on a 120-Isosceles Triangular Billiards Table

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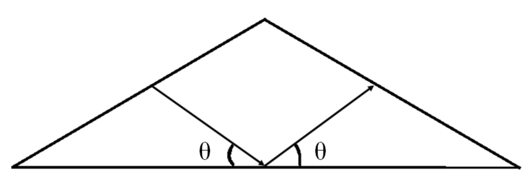
The Problem

Find and classify the periodic orbits on a 120-isosceles triangular billiard table.

Assumptions

1. A billiard ball bounce follows the same rule as a reflection: Angle of incidence = Angle of reflection



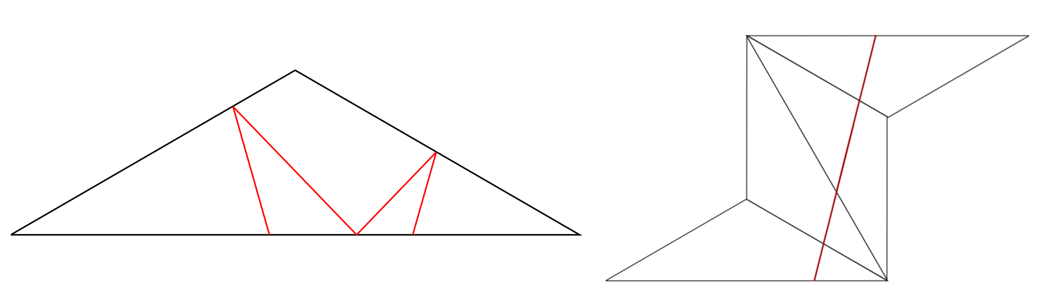
2. A billiard ball stops if it hits a vertex.

Definitions

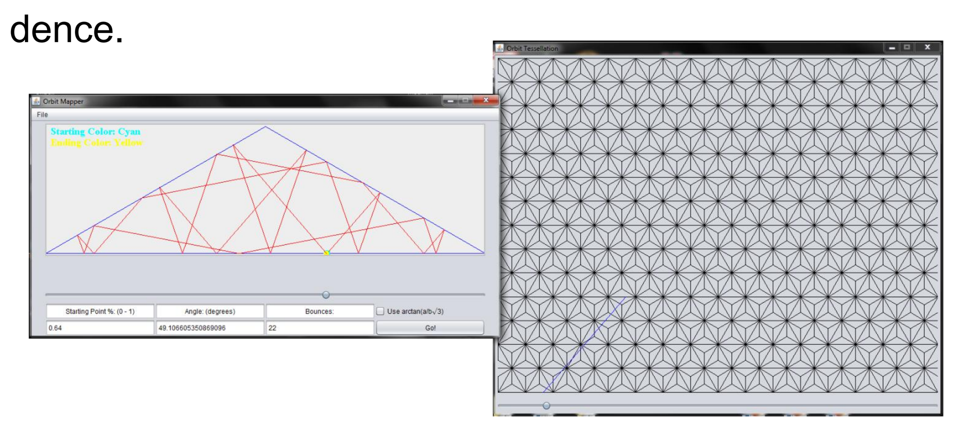
1. The orbit of a billiard ball is the trajectory it follows.
2. A singular orbit terminates at a vertex.
3. A periodic orbit eventually retraces itself.
4. The period of a periodic orbit is the number of bounces it makes until it starts to retrace itself.

Techniques of Exploration

1. We found it easier to analyze the path of the billiard ball by reflecting the triangle about the side of impact. In the equilateral case we were able to construct a tessellation, the same can be done with the 120-isosceles case



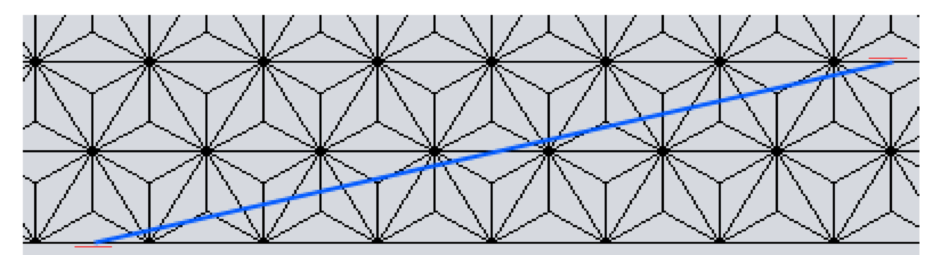
2. We used Josh Pavoncello's Orbit Mapper program to generate orbits with a given initial angle and initial point of incidence.



(22 bounce orbit using the Orbit Mapper program)

Facts About Orbits

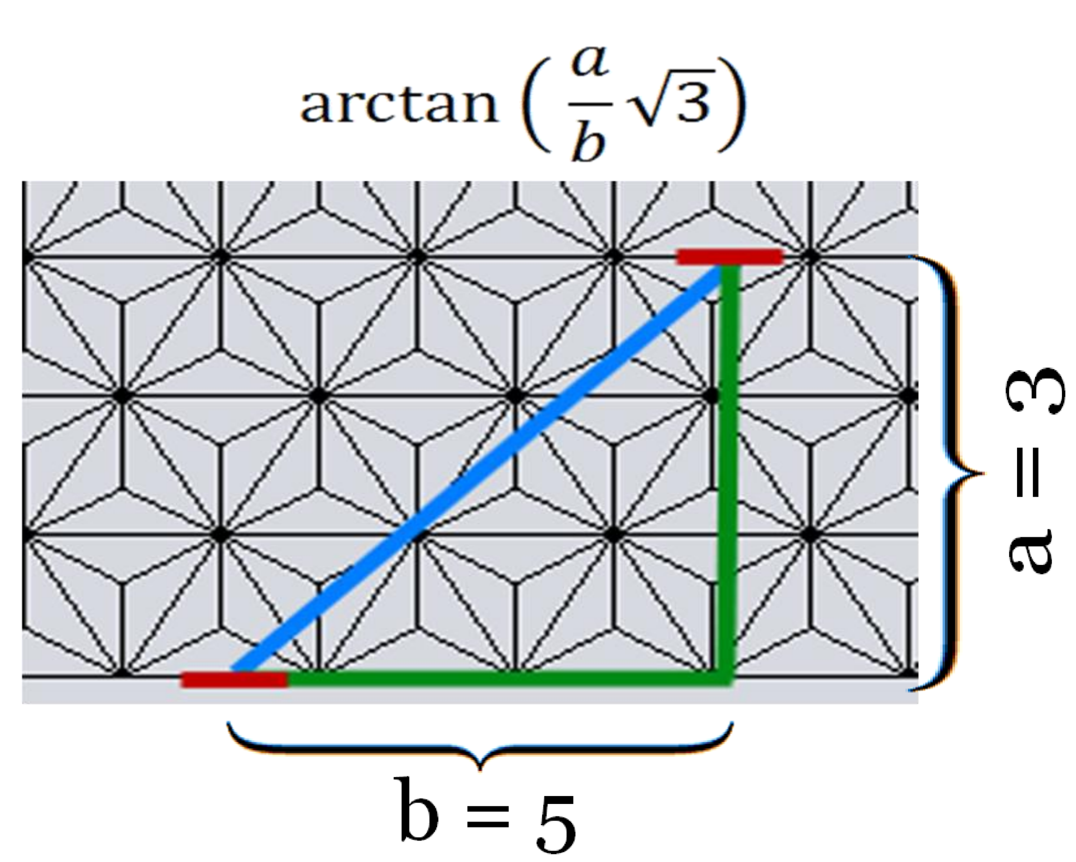
Theorem 1: *If the initial point of a periodic orbit is on a horizontal edge of the tessellation, so is its terminal point.*



Orbit of 36 bounces

Proof: Assume that there exists a periodic orbit that does not terminate on a horizontal. Then the period must end on one of the two sides of the equilateral triangles that make up the tessellation. Also the angle going through these sides must be equal to the initial angle going through the horizontal. Computing the relation between the initial angle and this angle going through the sides and setting it equal to the initial angle gives the result that for the an orbit to end on a non-horizontal side the initial angle must be either 30 or 60 degrees. But when the initial angle is 30 an orbit of period 8 that always ends on the horizontal is produced, so the initial angle must be 60. But when the initial angle is 60, one of two orbits (of periods 4 and 10 respectively) is produced and both of those orbits always terminate on a horizontal. Therefore, no periodic orbit ends on a non-horizontal side and thus all periodic orbits must terminate on a horizontal. Also every orbit must hit the side of the triangle opposite the 120 degree angle (which we are considering parallel to the horizontal) therefore we can adjust the starting point of any orbit onto the horizontal.

Theorem 2: *An angle that generates a periodic orbit will be 90° or in the form*



Double and Single Period Orbits

Let $p(a,b)$ be a function which outputs the period(s) of a given angle $\arctan(\frac{a}{b}\sqrt{3})$ where $a, b \in \mathbb{N}$, $a \geq b$, and a, b are relatively prime. Then

$$p(a,b) = \begin{cases} Single : 8a + \frac{4b}{3} & \text{if } b \equiv 0 \ (mod\ 6) \\ Lower : 8a + \frac{4b+2}{3};\ Upper : 16a + \frac{8b-2}{3} & \text{if } b \equiv 1 \ (mod\ 6)\ and\ 2 \mid a \\ Lower : 4a + \frac{2b-2}{3};\ Upper : 8a + \frac{4b+2}{3} & \text{if } b \equiv 1 \ (mod\ 6)\ and\ 2 \nmid a \\ Lower : 8a + \frac{4b-2}{3};\ Upper : 16a + \frac{8b+2}{3} & \text{if } b \equiv 2 \ (mod\ 6) \\ Single : 8a + \frac{4b}{3} & \text{if } b \equiv 3 \ (mod\ 6)\ and\ 2 \mid a \\ Single : 4a + \frac{2b}{3} & \text{if } b \equiv 3 \ (mod\ 6)\ and\ 2 \nmid a \\ Lower : 8a + \frac{4b+2}{3};\ Upper : 16a + \frac{8b-2}{3} & \text{if } b \equiv 4 \ (mod\ 6) \\ Lower : 8a + \frac{4b-2}{3};\ Upper : 16a + \frac{8b+2}{3} & \text{if } b \equiv 5 \ (mod\ 6)\ and\ 2 \mid a \\ Lower : 4a + \frac{2b+2}{3};\ Upper : 8a + \frac{4b-2}{3} & \text{if } b \equiv 5 \ (mod\ 6)\ and\ 2 \nmid a \end{cases}$$

Proof: Denote the length of one of the bases of the 30-30-120 triangles along the horizontal by 2d, then the vertical distance between two horizontals is √(3)*d. Now consider the path of any periodic orbit on the plane tessellation. From Theorem 1 we know that this orbit must start and end on a horizontal. Now break the orbit down into x and y components, the y component must be some integer multiple of √(3)*d (y comp = √(3)*dm for some integer m) and the x component must be some integer multiple of 2d (x comp = 2dn for some integer n). Now the initial angle, lets call it θ, can be calculated using tan θ = (y-comp/ x-comp) (assuming the x component is not zero in which case θ would be 90 degrees). Solving the above equation for θ produces: θ = arctan (y-comp/ x-comp).

But (y-comp/ x-comp) = √(3)*m/(2n) and since m and n are integers (m/2n) is a rational number, lets label that rational number q. Then substituting back we have θ = arctan (√(3)q) for some rational number q.

Theorem 3: *Let θ = arctan (√(3)a/b) be the initial angle that generates periodic orbits, where a and b are relatively prime positive integers. Then*

(1) If b is divisible by 3, then there is exactly one possible period.

(2) If b is not divisible by 3, then there are exactly two distinct periods. Furthermore, if the two periods are m and n with m < n, then n = 2m + 2 or n = 2m − 2.

Where We Are Headed

We still have yet to prove the formulas for calculating the periods given an initial angle. As well we have to work on classifying all of the orbits and counting the orbits of a certain period. We are very confident on the findings so far.