These undergraduate demonstrated that the ten conjectured surface-area minimizing double bubbles configurations in Figure 1 physically exist by constructing a soap film representation of each type in a plexiglass box.

Double Bubble Experiments in the Three-Torus

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All of us have blown soap bubbles and admired their perfectly spherical shape. Soap bubbles are spherical because spheres minimize the surface area required to enclose a given volume. In 2000, mathematicians Michael Hutchings, Frank Morgan, Manuel Ritoré, and Antonio Ros announced a proof that the surface area minimizing configuration enclosing two volumes consists of two spheres joined together to form the "standard double bubble" pictured in the upper left corner of Figure 1.

Imagine yourself standing in a magical room whose ceiling and floor as well as opposite walls are mysteriously connected so that when you look up you see your feet; when you look to your left you see your right; and when you look straight ahead you see your back. This imaginary room is a four-dimensional mathematical object called a "three-torus." Now imagine yourself blowing bubbles in a three-torus.When your bubbles are small relative to the size of the room, you see the standard bubble. But when your bubbles are large enough to collide with one or more walls, new and interesting configurations appear.

In 2001, Miguel Carrión Álvarez, Joseph Corneli, Genevieve Walsh, and Shabnam Beheshti (participants at the Clay Mathematics Institute Summer School on the Global Theory of Minimal Surfaces, held at the Mathematical Sciences Research Institute in Berkeley, California) proposed many possible surface-area minimizing double bubble configurations in a 3-torus. Using calculations and computer simulations, these participants eliminated all proposed configurations except the ten distinct topological types pictured in Figure 1. But whether or not every possible type is one of these ten is an open question.

During the spring of 2007, we participated in an undergraduate research seminar at Millersville University directed by professors Frank Morgan of Williams College and Ron Umble of Millersville University. We demonstrated that the ten configurations in Figure 1 physically exist by constructing a soap film representation of each type in a plexiglass box (see

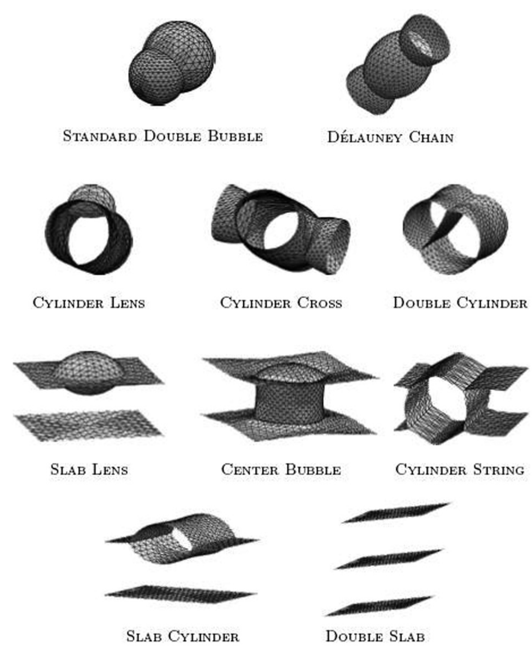


Figure 1. The ten conjectured surface-area minimizers (Image from article by Alvarez et al, Double Bubbles in the Three Torus, *Experimental Math*, 2003. Reproduced with permission.)

Figures 3–12). One obtains a three-torus by identifying opposite walls of our box as indicated in Figure 2; these identifications make opposite walls the same.

To create our soap films, we used a mixture of one part JOY dish detergent, two parts glycerin, and three parts water. We began by coating the walls of the box with this solution. This made it easier to transfer soap bubbles from our wand to the walls of the box. In some cases, we created a portion of the surface then applied reflections and rotations to obtain the complete surface. It is conceivable that some of the surfaces created in this way are physically unstable, and in several cases stability is an open question. The following basic bubbles appear in our double bubble constructions:

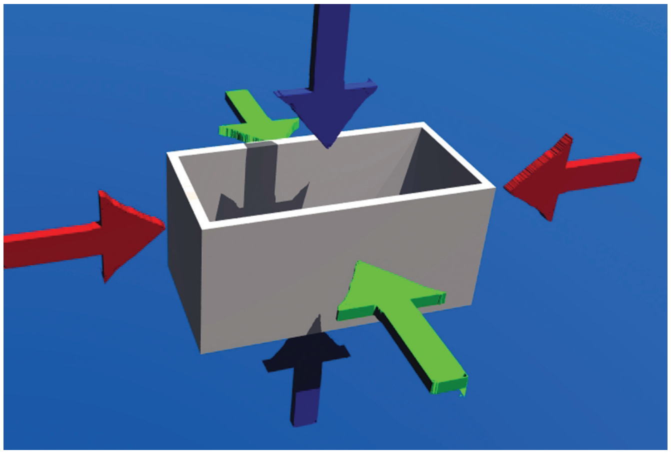


Figure 2. A 3-torus obtained by identifying opposite sides of a box.

Standard Bubble

To blow the standard bubble, dip the wand into the soap solution and blow gently into the wand until a bubble is formed. Place the bubble on the bottom of the box, being careful not to let the bubble touch the sides of the box. This gives half of the standard bubble. To complete the construction, reflect the half-bubble in the bottom of the box.

Half Cylinder

To blow the half cylinder, dip the wand into the soap solution and blow, holding the bubble about a quarter of the way into the box. Continue blowing, and allow the bubble to grow until it touches a wall of the box. Once this happens, slowly move the wand toward the opposite wall until the bubble touches it and a half cylinder is formed. Note that one can obtain the cylinder by reflecting in the bottom of the box.

Quarter Cylinder

To blow the quarter cylinder, dip the wand into the soap solution. Starting in one corner of the box, blow a bubble that encompasses that corner. Continue blowing while slowly moving the wand toward an adjacent corner of the box. Continue until the bubble encompasses that corner as well and a quarter cylinder is formed. This time a complete cylinder is obtained by two reflections–once in the bottom and once in the side of the box touching the bubble.

Slab

To blow a slab, dip the wand into the soap solution and begin blowing with the wand positioned near the center of the box. Continue blowing until the bubble touches all four walls of the box. Release the wand, and obtain the standard slab.

Each set of instructions below describes how to create one of the ten types in Figure 1 and is accompanied by a photograph of our soap film representation.

Standard Double Bubble

To create the standard double bubble in Figure 3, invert the box to prevent the bubble from contacting the walls. Blow a half bubble on the base of the box and adjoin another half bubble to the first. Form the standard double bubble by reflecting these two half bubbles in the base of the box.

Figure 3. A standard double bubble on the base of an inverted box.

Delaunay Chain

To create the Delaunay chain in Figure 4, blow a half bubble in the middle of a narrow box. Blow a quarter bubble on each end of the box and allow them to merge with the half bubble. When this configuration is reflected in the base, we obtain the Delaunay chain.

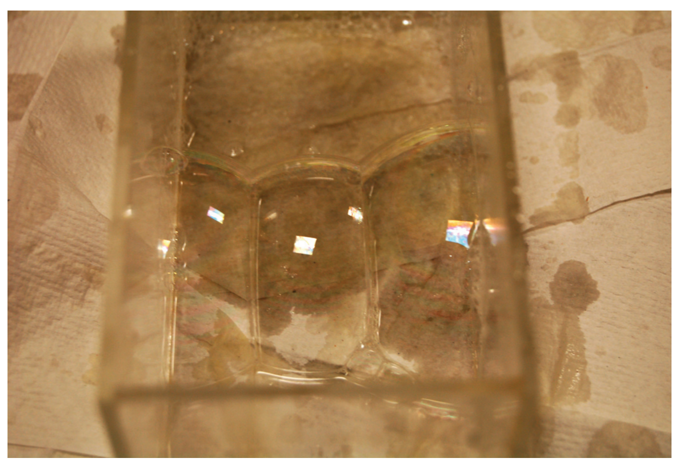


Figure 4. A Delaunay chain with one bubble between two halves of the second bubble; halves are joined at the walls.

Cylinder Lens

To create the cylinder lens in Figure 5, blow a half cylinder on the bottom of the box and reflect it to obtain a complete cylinder. Next, blow a bubble near the top half of the cylinder, allowing the half cylinder and the bubble to touch. Release the wand and obtain the cylinder lens.

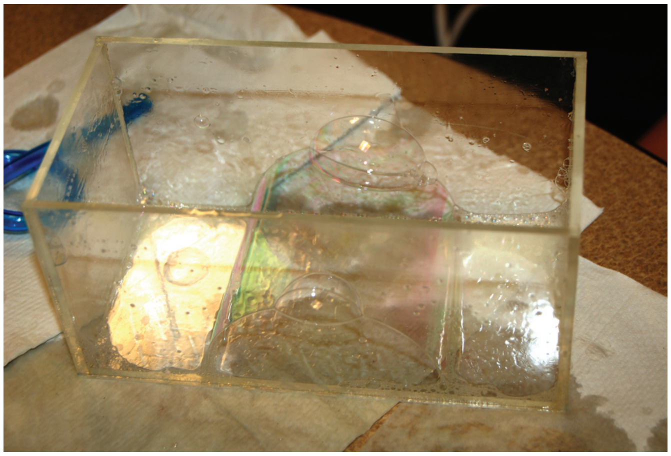


Figure 5. A cylinder lens.

Cylinder Cross

To create the cylinder cross in Figures 6(a) and 6(b), blow a quarter cylinder along the long side of the box. Then blow a half cylinder so that the half cylinder intersects the quarter cylinder perpendicularly. Reflecting across the base yields the cylinder cross.

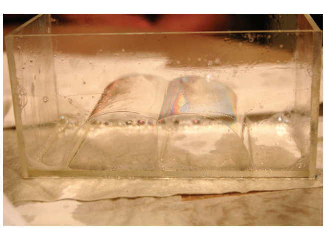
(6a)

(6b)

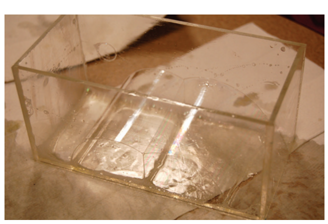
Figure 6. A cylinder cross. Two halves of the big cylinder at the left and right are joined by a thin cylinder at bottom center.

Double Cylinder

To create the double cylinder in Figures 7(a) and 7(b), blow a half cylinder that terminates on opposite walls. Blow a second half cylinder parallel to and touching the first that also terminates on the same opposite walls. Reflecting across the base yields the double cylinder.



(7a)



(7b)

Figure 7. A double cylinder.

Slab Lens

To create the slab lens in Figure 8, blow a slab and place a bubble on top of the slab.

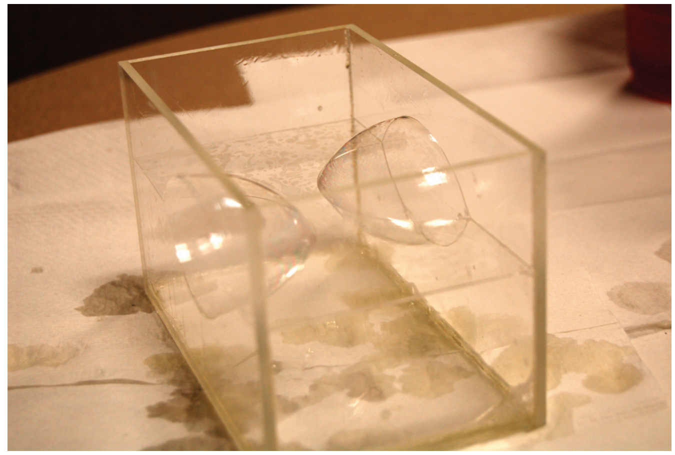


Figure 8. A slab lens.

Center Bubble

To create the center bubble in Figure 9, blow a slab. Obtain half of the central cylinder by placing the wand directly on top of the slab and near the middle of one long side of the box. Blow a new bubble that touches the bottom of the box, the side, and the slab. Reflecting across the base and the side touching the half cylinder yields the center bubble.

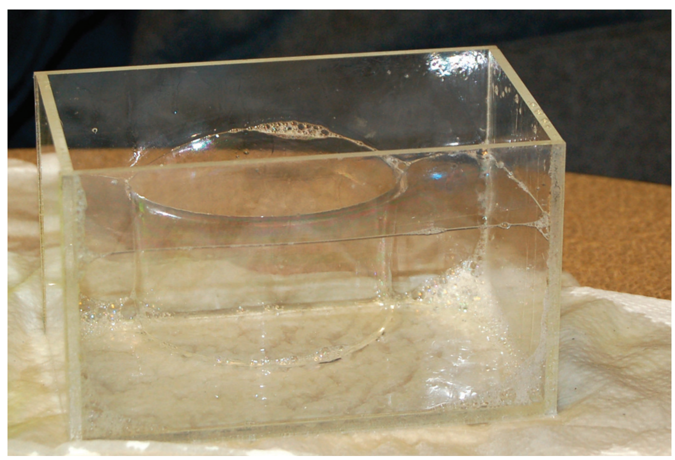
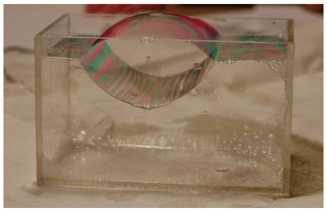


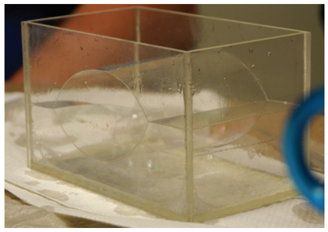
Figure 9. A center bubble.

Slab Cylinder

To create the slab cylinder pictured in Figures 10(a) and 10(b), blow a slab, then blow a bubble touching one wall of the box above the slab. Drag this bubble across the slab until it touches the opposite wall creating a cylinder. When this cylinder merges with the slab we obtain the slab cylinder.



(10a)



(10b)

Figure 10. The slab cylinder.

Cylinder String

To create the cylinder string in Figure 11, blow a half cylinder terminating on opposite walls of the box. Next, blow two quarter cylinders parallel to and touching the half cylinder, one on either side, also terminating on the same opposite walls. Reflecting in the base yields the cylinder string.

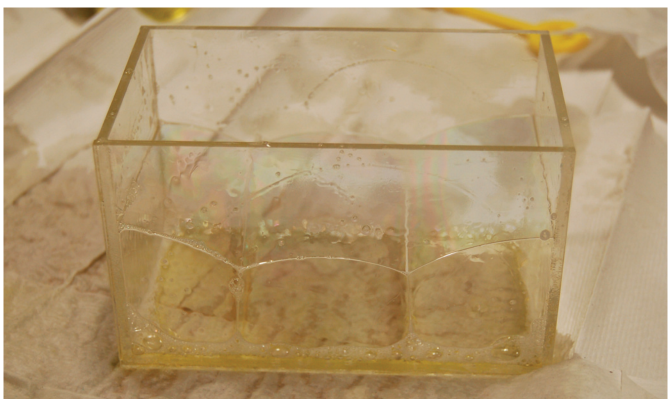
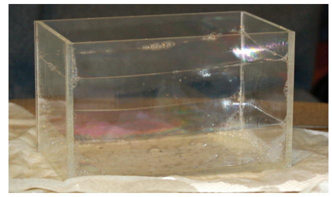


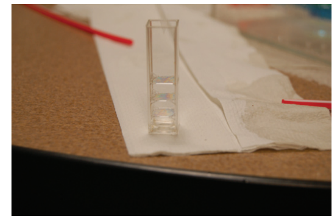
Figure 11. A cylinder string.

Double Slab

To create the double slab in Figures 12(a) and 12(b), blow a slab. Coat a straw with soap solution and gently push it through the slab. Next, blow two bubbles below the slab that merge together and touch all four sides of the box. These bubbles are separated by an unwanted membrane. Again, coat a straw with the soap solution and gently push it through the slab and one of the two bubbles until it touches the unwanted membrane. Blow hard until this membrane ruptures and the two bubbles merge into one slab. This yields two parallel slabs; repeat this construction to obtain a third parallel slab.



(12a)



(12b)

Figure 12. A double slab.

Our physical soap film constructions in a plexiglass box model of a three-torus demonstrate that the ten conjectured surface area minimizing double bubbles pictured in Figure 1 exist, but whether or not this list is exhaustive remains an open question. Many of our realizations involved reflecting parts of a bubble across one or more sides of the box. In several such cases, questions of physical stability also remain open.

Acknowledgments

The authors are indebted to professors Ron Umble of Millersville University and Frank Morgan of Williams College.

Further Reading

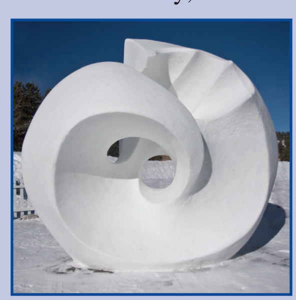
M. Carrión, J. Corneli, G. Walsh, S. Beheshti. “Double Bubbles in the Three-Torus,” *Experimental Math* 12 (2003) 79–89.

M. Hutchings, F. Morgan, M. Ritoré, A. Ros. “Proof of the Double Bubble Conjecture,” *Ann. Math.* 155 (2002) 459–489.

Cold Hands, Warm Heart

This snow sculpture, a design of David Chamberlain, was created at the annual competition in Breckenridge, CO, by Chamberlain, Dan Schwalbe, Rich and Beth Seeley, and Stan Wagon. More info at `http://stanwagon.com`.

Unlocking the secrets of the human heart is a subtle process. Cold hands, a reserved exterior, or unusual appearance can all disguise a warm heart, just as our sculpture's cool exterior masks the complexity of its inner heart.



Photos by Rich Seeley