AN A_∞-COALGEBRA STRUCTURE ON A CLOSED COMPACT SURFACE

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Abstract. Let P be an n-gon with $n \geq 3$. There is a formal combinatorial A_∞-coalgebra structure on cellular chains $C_*(P)$ with non-vanishing higher order structure when $n \geq 5$. If X_g is a closed compact surface of genus $g \geq 2$ and P_g is a polygonal decomposition, the quotient map $q : P_g \to X_g$ projects the formal A_∞-coalgebra structure on $C_*(P_g)$ to a quotient structure on $C_*(X_g)$, which persists to homology $H_*(X_g; \mathbb{Z}_2)$, whose operations are determined by the quotient map q, and whose higher order structure is non-trivial if and only if X_g is orientable or unorientable with $g \geq 3$. But whether or not the A_∞-coalgebra structure on homology observed here is topologically invariant is an open question.

To Tornike Kadeishvili on the occasion of his 70th birthday

1. Introduction

Let R be a commutative ring with unity and let P be an n-gon with $n \geq 3$. In this paper we construct a formal combinatorial A_∞-coalgebra structure on the cellular chains of P, denoted by $C_*(P)$, which is the graded R-module generated by the vertices, edges, and region of P. For an application, let X_g be a closed compact surface of genus $g \geq 2$ and let P_g be a polygonal decomposition. The quotient map $q : P_g \to X_g$ sends the formal A_∞-coalgebra structure on $C_*(P_g)$ to a quotient structure on $C_*(X_g)$, which persists to homology $H_*(X_g; \mathbb{Z}_2)$, whose operations are determined by the quotient map q, and whose higher order structure is non-trivial if and only if X_g is orientable or unorientable with $g \geq 3$.

An A_∞-coalgebra is the linear dual of an A_∞-algebra defined by J. Stasheff [6] in the setting of base pointed loop spaces. As motivation, we begin with a brief description of A_∞-algebras.

Let S be a surface embedded in $\mathbb{R}^3$ and let $*$ be some specified base point on S. A *base pointed loop on* S is a continuous map $\alpha : I \to S$ such that $\alpha(0) = \alpha(1) = *$. Let ΩS denote the space of all base pointed loops on S. Given $\alpha, \beta \in \Omega S$, define their *product* $\alpha \cdot \beta \in \Omega S$ to be

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & t \in [0, \frac{1}{2}] \\ \beta(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

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[1] The main result (Theorem 4) was proved by the first author in his senior honors thesis [4].

A *homotopy from* α *to* β is a continuous map $H : I \to \Omega S$ such that $H(0) = \alpha$ and $H(1) = \beta$. Thus $\{H(s) : s \in I\}$ is a 1-parameter family of loops that continuously deforms α to β.

Let $\alpha, \beta, \gamma \in \Omega S$. Although $(\alpha \cdot \beta) \cdot \gamma \neq \alpha \cdot (\beta \cdot \gamma)$, the loops $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ are homotopic via a linear change of parameter homotopy H. Let $\mathbf{1} : \Omega S \to \Omega S$ be the identity map and define $m_2 : \Omega S \otimes \Omega S \to \Omega S$ by $m_2(\alpha \otimes \beta) = \alpha \cdot \beta$. Then $m_2(m_2 \otimes \mathbf{1})(\alpha \otimes \beta \otimes \gamma) = (\alpha \cdot \beta) \cdot \gamma$ and $m_2(\mathbf{1} \otimes m_2)(\alpha \otimes \beta \otimes \gamma) = \alpha \cdot (\beta \cdot \gamma)$. Consider $m_2(m_2 \otimes \mathbf{1}), m_2(\mathbf{1} \otimes m_2) : \Omega S^{\otimes 3} \to \Omega S$ and think of the homotopy H from $(\alpha \cdot \beta) \cdot \gamma$ to $\alpha \cdot (\beta \cdot \gamma)$ as a 3-ary operation $m_3 : \Omega S^{\otimes 3} \to \Omega S$. Identify m_3 with the interval $[0, 1]$, its endpoint 0 with $m_2(m_2 \otimes \mathbf{1})$, and its endpoint 1 with $m_2(\mathbf{1} \otimes m_2)$. Then the boundary $\partial m_3 = m_2(\mathbf{1} \otimes m_2) - m_2(m_2 \otimes \mathbf{1})$ and the parameter space $[0, 1]$ identified with m_3 is called the associahedron K_3. Thus K_3 controls homotopy associativity in three variables.

In a similar way, homotopy associativity in four variables is controlled by the associahedron K_4, which is a pentagon. The vertices of K_4 are identified with the five ways one can parenthesize four variables, its edges are identified with the homotopies that preform a single shift of parentheses, and its 2-dimensional region is identified with a 4-ary operation $m_4 : \Omega S^{\otimes 4} \to \Omega S$. Thus $\partial m_4 = m_2(m_3 \otimes \mathbf{1}) - m_3(m_2 \otimes \mathbf{1} \otimes \mathbf{1}) + m_3(\mathbf{1} \otimes m_2 \otimes \mathbf{1}) - m_3(\mathbf{1} \otimes \mathbf{1} \otimes m_2) + m_2(\mathbf{1} \otimes m_3)$. In general, homotopy associativity in n variables is controlled by the associahedron K_n, which is an $n-2$ dimensional polytope whose vertices are identified with the various ways to parenthesize n variables. While associahedra are independently interesting geometric objects, they also organize the data in the definition of an A_∞-(co)algebra.

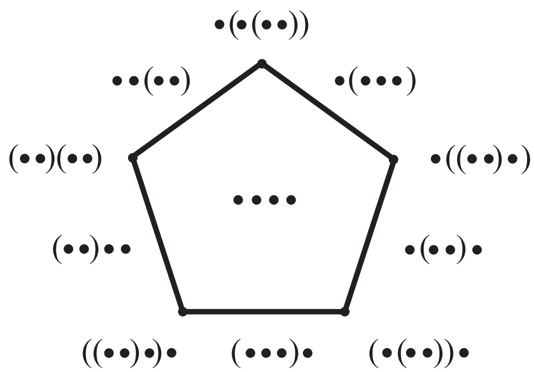

Figure 1. The associahedron K_4.

But before we can define an A_∞-(co)algebra, we need some preliminaries. A *differential graded (d.g.) R-module* is a graded R-module $V = \oplus_{i \geq 0} V_i$ equipped with a differential operator $\partial : V_* \to V_{*-1}$ such that $\partial \circ \partial = 0$. Let (V, ∂_V) and (W, ∂_W) be d.g. R-modules. A linear map $f : V \to W$ has *degree p* if $f : V_i \to W_{i+p}$; the map f is a *chain map* if $f \circ \partial_V = (-1)^p \partial_W \circ f$. Denote the degree f by $|f|$ and the R-module of all linear maps of degree p by $Hom_p(V, W)$.

Proposition 1. *$Hom_*(V, W)$ is a d.g. R-module with differential δ given by $\delta(f) = f \circ \partial_V - (-1)^{|f|} \partial_W \circ f$.*

Proof. The proof is straight-forward and omitted. $\square$

Note that f is a chain map if and only if $\delta(f)=0$. Now $m_2 \in Hom_*(C_*(P)^{\otimes 2}, C_*(P))$ and $m_3 \in Hom_*(C_*(P)^{\otimes 3}, C_*(P))$. Since $|m_3|=1$ we have

$$\delta(m_3) = m_3 \circ \partial^{\otimes 3} - (-1)^1 \partial \circ m_3 = \partial \circ m_3 = m_2(\mathbf{1} \otimes m_2) - m_2(m_2 \otimes \mathbf{1}),$$

where $m_3 \circ \partial^{\otimes 3} = 0$ because loops have empty boundary. Then $\delta(m_3)$ measures the *deviation of m_2 from associativity*, and in certain situations we can express this deviation in terms of a degree 0 chain map $\alpha_3 : C_*(K_3) \to Hom_*(C_*(P)^{\otimes 3}, C_*(P))$. Let θ_n denote the top dimensional cell of K_n.

Definition 2. *Let (V,∂) be a d.g. R-module. For each $n \geq 2$, choose a map $\alpha_n : C_*(K_n) \to Hom(V^{\otimes n}, V)$ of degree 0, and let $m_n = \alpha_n(\theta_n)$. Then $(V, \partial, m_2, m_3, \dots)$ is an A_∞-**algebra** if $\delta\alpha_n = \alpha_n \partial$ for each $n \geq 2$.*

The definition of an A_∞-coalgebra mirrors the definition of an A_∞-algebra.

Definition 3. *Let (V,∂) be a d.g R-module. For each $n \geq 2$, choose a map $\alpha_n : C_*(K_n) \to Hom(V, V^{\otimes n})$ of degree 0, and let $\Delta_n = \alpha_n(\theta_n)$. Then $(V, \partial, \Delta_2, \Delta_3, \ldots)$ is an A_∞-**coalgebra** if $\delta\alpha_n = \alpha_n \partial$ for each $n \geq 2$.*

Evaluating both sides of the equation in Definition 3 at θ_n produces the classical structure relation

$$\Delta_n \partial - (-1)^{n-2} \sum_{i=0}^{n-1} \left(\mathbf{1}^{\otimes i} \otimes \partial \otimes \mathbf{1}^{\otimes n-i-1} \right) \Delta_n$$

$$= \sum_{i=1}^{n-2} \sum_{j=0}^{n-i-1} (-1)^{i(j+n+1)} \left(\mathbf{1}^{\otimes j} \otimes \Delta_{i+1} \otimes \mathbf{1}^{\otimes n-i-j-1} \right) \Delta_{n-i}, \tag{1}$$

which expresses Δ_n as a chain homotopy among the quadratic compositions encoded by the codimension 1 cells of K_n (see Figure 2). The sign $(-1)^{l(n+i+1)}$ is the combinatorial sign derived by Saneblidze and Umble in [5]. Note that when $n=2$, Relation 1 has the form $\Delta_2 \partial = (\partial \otimes \mathbf{1} + \mathbf{1} \otimes \partial) \Delta_2$, which is dual to the Leibniz Rule $dm = m(d \otimes \mathbf{1} + \mathbf{1} \otimes d)$ in calculus and says that ∂ is a coderivation of Δ_2.

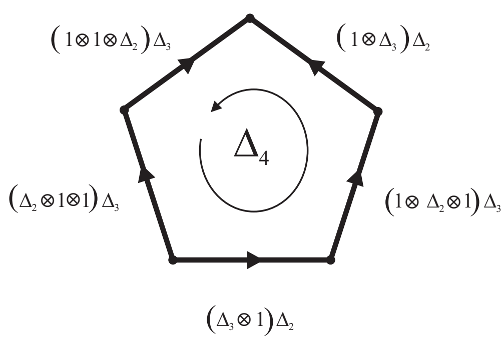

Figure 2. The quadratic compositions encoded by the codim 1 cells of K_4.

2. Statement of the Main Result

Let P be a counterclockwise oriented n-gon, $n \geq 3$. Label the vertices $v_1, v_2, \ldots v_n$ and the edges $e_1, e_2, \ldots e_n$. Define v_1 to be the *initial vertex* and v_n to be the *terminal vertex*, and direct the edges from v_1 to v_n. This assignment partially orders the vertices as indicated in Figure 3. Edges whose direction is consistent with orientation are positive. Let $\partial : C_*(P) \to C_{*-1}(P)$ be the boundary operator induced by geometric boundary. Then $\partial(v_i) = 0$, $\partial(e_i) = v_{i+1} - v_i$ if $i < n$, $\partial(e_n) = v_n - v_1$, and $\partial(p) = e_1 + e_2 + \cdots + e_{n-1} - e_n$.

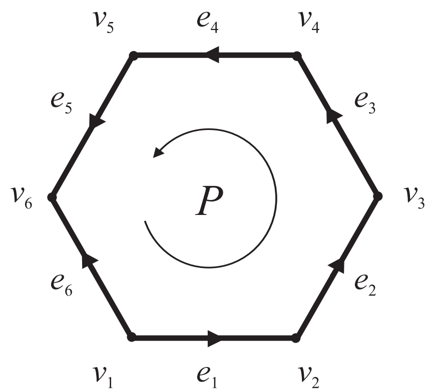

Figure 3. An n-gon for $n = 6$

We will use the diagonal approximation $\Delta_2 : C_*(P) \to C_*(P) \otimes C_*(P)$ defined by D. Kravatz in [2] and given by

$$\begin{aligned}
\Delta_2(P) &= v_1 \otimes P + P \otimes v_n + \sum_{0<i_1<i_2<n} e_{i_1} \otimes e_{i_2}, \\
\Delta_2(e_i) &= v_i \otimes e_i + e_i \otimes v_{i+1} \text{ if } i < n, \\
\Delta_2(e_n) &= v_1 \otimes e_n + e_n \otimes v_n, \\
\Delta_2(v_i) &= v_i \otimes v_i.
\end{aligned}$$

A more general exposition of Kravatz's diagonal appears in [1]. For $k > 2$, define the k-ary A_∞-coalgebra operation $\Delta_k : C_*(P) \to C_*(P)^{\otimes k}$ by

$$\Delta_k(\sigma) = \begin{cases} \sum_{0<i_1<i_2<\cdots<i_k<n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, & \text{if } \sigma = P \\ 0, & \text{otherwise.} \end{cases}$$

Then by definition, $\Delta_k = 0$ for all $k \geq n$. We can now state our main result.

Theorem 4. *Let P be an n-gon. The structure operations $\{\Delta_k\}_{2\leq k<n}$ defined above impose an A_∞-coalgebra structure on $(C_*(P), \partial)$. Furthermore, Δ_k vanishes for all $k \geq n$.*

3. Proof of the Main Result

We must verify Relation 1 for all $k \geq 2$. When $k = 2$, verification is easy and left to the reader. We first verify Relation 1 for all $k > 3$, then consider the special case $k = 3$. To simplify notation we establish following notational devices:

- The symbol $\partial\Delta_k(P)$ denotes $\sum_{i=0}^{k-1}(\mathbf{1}^{\otimes i}\otimes\partial \otimes \mathbf{1}^{\otimes k-i-1})\Delta_k(P)$.
- The symbol $\Delta_j\Delta_k(P)$ denotes $\sum_{i=0}^{k-1}(\mathbf{1}^{\otimes i}\otimes\Delta_j \otimes \mathbf{1}^{\otimes k-i-1})\Delta_k(P)$.

The fact that Δ_j and Δ_k vanish on edges and vertices when $j, k \geq 3$ implies:

Proposition 5. $\Delta_j\Delta_k(P) = \partial\Delta_k(P) = 0$ *whenever* $i, j \geq 3$.

In view of Proposition 5, all non-vanishing terms in Relation 1 apply some Δ_k to the 2-cell P. Therefore (up to sign) Relation 1 reduces to

$$(\Delta_{k-1}\Delta_2 + \Delta_2\Delta_{k-1} + \partial\Delta_k)(P) = 0. \tag{2}$$

The signs in Relation 2 follow from the *Sign Commutation Rule*: *If an object of degree p passes an object of degree q, affix the sign* $(-1)^{pq}$. First, consider a term $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}$ of $\Delta_k(P)$. Since $|\partial| = -1$, multiplying by the sign in Relation 1 and simplifying gives

$$(-1)^{k-2}\left(\mathbf{1}^{\otimes i-1} \otimes \partial \otimes \mathbf{1}^{\otimes k-i}\right)(e_{j_1} \otimes \cdots \otimes e_{j_k}) = (-1)^{i+k+1}\, e_{j_1} \otimes \cdots \otimes \partial e_{j_i} \otimes \cdots \otimes e_{j_k}.$$

Second, since $|\Delta_2| = 0$, the sign of $\Delta_2\Delta_{k-1}$ is the sign $(-1)^{i+k}$, which is opposite the sign in Relation 1, where Δ_2 is applied in the i^{th} position. Third, the signs of the terms $(\Delta_{k-1} \otimes \mathbf{1})\Delta_2$ and $(\mathbf{1} \otimes \Delta_{k-1})\Delta_2$ given by Relation 1 simplify to -1 and $(-1)^{k+1}$, respectively. Since $|\Delta_{k-1}| = k-3$, the Sign Commutation Rule can only introduce a sign when $\mathbf{1} \otimes \Delta_{k-1}$ is applied to a pair of edges. However, this particular situation never occurs in our proof.

Lemma 6. *All non-vanishing terms contain exactly one tensor factor v_j. Furthermore, if $e_i \otimes v_j$ or $v_j \otimes e_k$ appears within some term, then $i < j$ or $j \leq k$.*

Proof. **Case 1:** Consider $\partial\Delta_k(P)$. Note that $\Delta_k(P)$ will either vanish if $k \geq n$, or it will produce terms of the form $e_{i_1} \otimes \cdots \otimes e_{i_k}$, where $0 < i_1 < \cdots < i_k < n$. Applying ∂ to any e_i will create two new terms by replacing it with v_i or v_{i+1}. Since the subscript i is between the subscripts on either side, either choice gives the desired result.

Case 2: Consider $\Delta_{k-1}\Delta_2(P)$. Since Δ_{k-1} acts non-trivially only on P, the terms of interest produced by Δ_2 are $v_1 \otimes P$ and $P \otimes v_n$. When Δ_{k-1} acts on the P, it produces a term of the form $e_{i_1} \otimes \cdots \otimes e_{i_{k-1}}$, where $0 < i_1 < \cdots < i_{k-1} < n$. Since $0 < i < n$ for all i, all terms either begin with v_1 or end with v_n and have the desired form.

Case 3: Consider $\Delta_2\Delta_{k-1}(P)$. Notice that $\Delta_{k-1}(P)$ produces terms of the form $e_{i_1} \otimes \cdots \otimes e_{i_{k-1}}$, where $0 < i_1 < \cdots < i_{k-1} < n$. When Δ_2 is applied to an e_i factor, it has the effect of either inserting v_i to the left, or a v_{i+1} to the right of the e_i. Since the subscript i is between the subscripts on either side, either choice gives the desired result. □

This information allows us to classify the terms in Relation 2 relative to the position of v_i with respect to e_{i-1} and e_i. We say that v_i is *left adjacent* if e_{i-1} is immediately to its left; v_i is *right adjacent* if e_i is immediately to its right.

Definition 7. *A term in which v_i is both left and right adjacent is* ***doubly attached****. A term in which v_i is either left adjacent or right adjacent but not both is* ***singly attached****. A term in which v_i is neither left adjacent nor right adjacent is* ***unattached****.*

We can sub-classify within these sets as follows:

Definition 8. *A term that begins with v_1 is* ***extreme minimal****. A term that ends with v_n is* ***extreme maximal****.*

Since doubly attached terms cannot be extreme, there are five classes of terms, and the proof reduces to showing that each class cancels itself in Relation 2. But before we can do this, we need some lemmas.

Lemma 9. *Let $2 < k < n$. If v_i appears in the i^{th} position of a term in $\partial\Delta_k(P)$, that term appears at most two times and only in the following ways:*

- ***Way 1:*** *The term will be generated once if and only if it is unattached on the left and is not minimally extreme; the term will have the sign $(-1)^{i+k+1}$.*
- ***Way 2:*** *The term will be generated once if and only if it is unattached on the right and is not maximally extreme; the term will have the sign $(-1)^{i+k}$.*

Proof. **Case 1:** Consider a term that contains $e_{i_1} \otimes v_{i_2} \otimes e_{i_3}$, where $0 < i_1 < i_2 \leq i_3 < n$. Since $\Delta_k(P)$ cannot generate v_{i_2}, it was produced by ∂. Since v_{i_2} appears between e_{i_1} and e_{i_3}, we know that ∂ was applied to a term of the form $e_{i_1} \otimes e_x \otimes e_{i_3}$ with $0 < i_1 < x < i_3 < n$. Furthermore, there are only two possible edges e_x that ∂ could have acted upon to generate v_{i_2}, namely when $x = i_2 - 1$ or $x = i_2$. Therefore, since there are only two possible values for x, there are at most two ways to generate any given term.

Suppose that $x = i_2 - 1$. Since $i_1 < x$, we have $i_1 < i_2 - 1$ forcing the original term to be unattached on the left. To see that all left unattached terms can be generated, simply note that the subscript of any left unattached term will satisfy the inequality $i_1 < i_2 - 1$ by definition. Then $\Delta_k(P)$ produces the term $e_{i_1} \otimes e_{i_2-1} \otimes e_{i_3}$, where $0 < i_1 < i_2 \leq i_3 < n$, and produces the desired term after ∂ is applied to e_{i_2-1}. Now applying ∂ this way produces a positive v_{i_2} in the same position in which ∂ was applied, and by the discussion above on the sign commutation rule, the term's final sign can be calculated to be $(-1)^{i+k+1}$. Similarly, suppose $x = i_2$. Then all right unattached terms, and only those terms, can be generated. Similar analysis reveals that the term's final sign is $(-1)^{i+k}$.

Case 2: Consider a term of the form $v_{i_2} \otimes e_{i_3} \otimes \cdots$, where $0 < i_2 \leq i_3 < n$. By an argument similar to the one above, ∂ must have been applied to a term of the form $e_x \otimes e_{i_3} \otimes \cdots$, where $x = i_2 - 1$ or $x = i_2$. The proof is similar to the case above and left to the reader; the only major difference is that when $x = i_2 - 1$ we cannot immediately conclude that all possible left unattached terms will be formed. This is because $x > 0$, and since $x = i_2 - 1$, we have $i_2 > 1$. Hence only non-minimally extreme left unattached terms can be formed, and an analysis similar to that in Case 1 shows that the terms produced have the sign $(-1)^{i+k+1}$.

Case 3: Consider a term of the form $\cdots \otimes e_{i_1} \otimes v_{i_2}$, where $0 < i_1 < i_2 < n$. This case is can be proved using an argument very similar to case 2, the details of which are left to the reader. □

Lemma 10. *Let $3 < k \leq n$. If v_i appears in the i^{th} position of a term in $\Delta_2\Delta_{k-1}(P)$, that term appears at most two times and only in the following ways:*

- ***Way 1:*** *The term will be generated once if and only if it is attached on the left; the term will have the sign $(-1)^{i+k+1}$.*
- ***Way 2:*** *The term will be generated once if and only if it is attached on the right; the term will have the sign $(-1)^{i+k}$.*

Proof. **Case 1:** Consider a term that contains $e_{i_1} \otimes v_{i_2} \otimes e_{i_3}$, where $0 < i_1 < i_2 \leq i_3 < n$. Since $\Delta_{k-1}(P)$ cannot produce v_{i_2}, it is produced by Δ_2, and because Δ_2 inserts a v next to the e to which it was applied, Δ_2 was applied to either e_{i_1} or e_{i_3}. Additionally, because of how Δ_2 is defined, the only way $\Delta_2(e_{i_1})$ can generate v_{i_2} is if $i_1 = i_2 - 1$, and the only way $\Delta_2(e_{i_3})$ can generate v_{i_2} is if $i_2 = i_3$. Therefore, there are at most two ways to create the desired term.

Now suppose Δ_2 was applied to e_{i_1} and that $i_1 = i_2 - 1$. Then by definition, the resulting term will be left attached. Furthermore, any given left attached term of the form $e_{i_2-1} \otimes v_{i_2} \otimes e_{i_3}$, where $0 < i_2 - 1 \leq i_3 < n$, can be created by simply applying Δ_2 to the term $e_{i_2-1} \otimes e_{i_3}$ generated by $\Delta_k(P)$. Now in order to create a term with v_{i_2} in the i^{th} position, we must apply Δ_2 to the $(i-1)^{st}$ position, and so by the discussion above on the sign commutation rule, the term's final sign can be calculated as $(-1)^{i+k+1}$. If we suppose that Δ_2 was applied to e_{i_3} and that $i_2 = i_3$, then a very similar analysis shows that all right attached terms of the form $e_{i_1} \otimes v_{i_2} \otimes e_{i_2}$, where $0 < i_2 < n$ are created and will have a final sign of $(-1)^{i+k}$.

Case 2: Consider a term of the form $v_{i_2} \otimes e_{i_3} \otimes \cdots$, where $0 < i_2 < i_3 < n$. By an argument similar to the one above, this term can only be generated if Δ_2 is applied to e_{i_3} and if $i_2 = i_3$. The same reasoning as in case one shows that all right attached terms, and only those terms, will be generated with sign $(-1)^{i+k}$.

Case 3: Consider a term of the form $\cdots \otimes e_{i_1} \otimes v_{i_2}$, where $0 < i_1 < i_2 < n$. This case is nearly identical to case 2, and we leave it to the reader to show that all left attached terms, and only those terms, can be generated with sign $(-1)^{i+k+1}$. $\square$

Proposition 11. *Let $2 < k < n$. If v_i appears in the i^{th} position of a term in $\partial\Delta_k(P)$, after cancellations the terms that remain are:*

- *All maximally extreme unattached terms with positive sign.*
- *All minimally extreme unattached terms with sign $(-1)^k$.*
- *All left attached terms with sign $(-1)^{i+k}$.*
- *All right attached terms with sign $(-1)^{i+k+1}$.*

Proof. The proof requires several straightforward applications of Lemma 9, some of which show that no doubly attached terms are generated, all unattached terms generated cancel, and all extreme terms generated are unattached. The details are left to the reader. $\square$

Proposition 12. *Let $3 < k \leq n$. Then $\Delta_{k-1}\Delta_2(P)$ contains:*

- *All minimally extreme terms with sign $(-1)^{k+1}$.*
- *All maximally extreme terms with negative sign.*

Proof. Since Δ_{k-1} only acts non-trivially on P, the only terms produced by Δ_2 that do not immediately vanish are $v_1 \otimes P + P \otimes v_n$. Additionally, note that since $k > 3$, Δ_{k-1} produces no primitive terms when it is applied by the definition of Δ_k for $k > 2$.

When Δ_{k-1} is applied to the first term, it generates terms of the form $v_1 \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots$, which are all minimally extreme. Since the sign commutation rule does not introduce anything new, we have that the term has the final sign $(-1)^{k+1}$.

When Δ_{k-1} is applied to the second term, it generates terms of the form $\cdots \otimes e_{i_{k-3}} \otimes e_{i_{k-2}} \otimes v_n$, which are maximally extreme. Since the sign commutation rule does not introduce anything new, we have that the sign will always be negative. $\square$

Proposition 13. *Let $3 < k \leq n$. If v_i appears in the i^{th} position of a term in $\Delta_2\Delta_{k-1}(P)$, after cancellations the terms that remain are:*

- *All maximally extreme singly attached terms with positive sign.*
- *All minimally extreme singly attached terms with sign $(-1)^k$.*
- *All left attached terms with sign $(-1)^{i+k+1}$.*
- *All right attached terms with sign $(-1)^{i+k}$.*

Proof. The proof requires several straightforward uses of Lemma 10 some of which show that no unattached terms are generated, all doubly attached terms cancel, and all extreme terms are singly attached. The details are left to the reader. □

Now when $k > 3$, all three propositions apply and we see that in Relation 2 the singly attached terms in $\partial\Delta_k(P)$ and $\Delta_2\Delta_{k-1}(P)$ cancel, the extreme unattached terms in $\partial\Delta_k(P)$ and $\Delta_{k-1}\Delta_2(P)$ cancel, and the extreme singly attached terms in $\Delta_{k-1}\Delta_2(P)$ and $\Delta_2\Delta_{k-1}(P)$ cancel. Therefore, all terms cancel and the relation is satisfied.

Having proved the theorem for $k > 3$, we consider the special case $k = 3$. We must show that:

$$[(\partial \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \partial \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \partial)\Delta_3 + (-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\Delta_2](P) = 0$$

Proposition 14. *The terms generated by $(\partial\Delta_3 + \Delta_2\Delta_2)(P)$ form three classes, which independently satisfy the conclusions of Propositions 11, 12, and 13.*

Proof. Note that Proposition 11 in the proof above still applies to $\partial\Delta_3(P)$ as before since it never used the restriction that $k \geq 3$. Therefore, we must show that $(\Delta_2 \otimes \mathbf{1} - \mathbf{1} \otimes \Delta_2)\Delta_2$ can be divided into parts which respectively satisfy the conclusions of Proposition 12 and Proposition 13. Now expanding we have

$$\begin{aligned}(-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\Delta_2(P) = (-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\,(v_1 \otimes P + P \otimes v_n)\\ (-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\sum_{i_1,i_2} e_{i_1} \otimes e_{i_2}.\end{aligned}$$

Notice that applications of $(-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\sum_{i_1,i_2} e_{i_1} \otimes e_{i_2}$ produce terms of the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{k-1}}$, where $0 < i_1 < i_2 < \cdots < i_{k-1} < n$, then applies Δ_2 to one of them. This is exactly the same set up as in Proposition 13, and since the proof of Proposition 13 only used the condition that $k \geq 3$ to ensure there are no primitive terms, we see that the proof applies to $(\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\sum_{i_1,i_2} e_{i_1} \otimes e_{i_2}$. Thus we obtain the same types of terms as given in Proposition 13.

Therefore, we must show that what remains behaves like the general case of $\Delta_{k-1}\Delta_2$ and produces all possible extreme terms. By expanding and some simple algebra, it is possible to show that

$$\begin{aligned}&(-\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\,(v_1 \otimes P + P \otimes v_n) =\\ &\qquad -\sum_{i_1,i_2} e_{i_1} \otimes e_{i_2} \otimes v_n + \sum_{i_1,i_2} v_1 \otimes e_{i_1} \otimes e_{i_2},\end{aligned}$$

which generates every possible extreme term and only extreme terms, all with the correct sign for $k = 3$. Therefore, $(\Delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_2)\,(v_1 \otimes P + P \otimes v_{n+1})$ satisfies the hypothesis of Proposition 12. □

Since the terms generated by $(\partial\Delta_3 + \Delta_2\Delta_2)(P)$ fall into classes that individually satisfy the conclusions of Proposition 11, Proposition 12, and Proposition 13, they cancel each other out as before. Therefore Δ_3 also satisfies Relation 1 and the proof of Theorem 4 is complete.

4. Generalization of Result

Until now, we have been working with an n-gon P whose initial and terminal vertices are adjacent. Our result extends to situations in which the initial and terminal vertices are non-adjacent. Suppose v_1 is the initial vertex and v_t is the terminal vertex, where $t > 1$. Define the generalized Δ_2' to be the same as in the introduction, except that

$$\Delta_2'(P) = v_1 \otimes P + P \otimes v_t + \sum_{0<i_1<i_2<t} e_{i_1} \otimes e_{i_2} - \sum_{n \geq i_1 > i_2 \geq t} e_{i_1} \otimes e_{i_2}.$$

Additionally, for $k > 2$, define the generalized k-ary A_∞-coalgebra operation Δ_k' by

$$\Delta_k'(\sigma) = \begin{cases} \displaystyle\sum_{0<i_1<i_2<\cdots<i_k<t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} & \\ \displaystyle\quad - \sum_{n \geq i_1 > i_2 > \cdots > i_k \geq t} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, & \text{if } \sigma = P \\ 0, & \text{otherwise.} \end{cases}$$

Then by definition, $\Delta_k' = 0$ for all $k \geq max\{t, n-t+2\}$. Now all that remains is to show that the operations $\{\Delta_n'\}_{n \geq 2}$ extend to this general setting.

Corollary 15. *Let P be an n-gon with initial vertex v_1 and terminal vertex v_t, where $t > 1$. The operations $\{\Delta_k'\}_{2 \leq k < max\{t, n-t+2\}}$ defined above impose an A_∞-coalgebra structure on $(C_*(P), \partial)$.*

Proof. Draw an additional edge from v_1 to v_t and denote it by e_0. Define P_1 to be the polygon with vertices $v_1, v_2, \ldots, v_t$ oriented counterclockwise and let P_2 to be the polygon with vertices $v_t, v_{t+1}, \ldots, v_n, v_1$ oriented counterclockwise as illustrated in Figure 4.

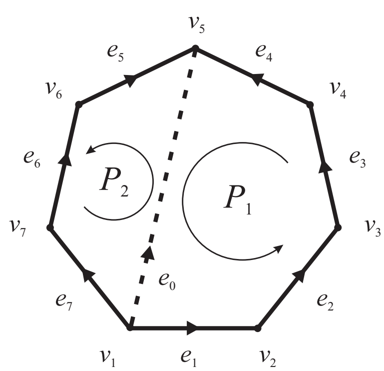

Figure 4. A 7-gon with $v_t = v_5$.

Then by the way edges are directed with respect to the orientation, we have $\partial(P_1) = e_1 + e_2 + \cdots + e_{t-1} - e_0$ and $\partial(P_2) = -e_t - e_{t+1} - \cdots - e_n + e_0$. Note that $\partial(P_1) + \partial(P_2) = \partial(P)$. Furthermore, define v_1 to be the initial vertex and v_t to be the terminal vertex in both P_1 and P_2; then P_1 and P_2 satisfy the hypothesis of Theorem 4 and it is straightforward algebra to show that $\Delta_2(P_1) + \Delta_2(P_2) = \Delta_2'(P)$ and $\Delta_k(P_1) + \Delta_k(P_2) = \Delta_k'(P)$ for $k > 2$. All that remains is to verify that Relation 1 holds on P for all $k \geq 2$. This can be done using the relations above and applying the main theorem to each part. The details are left to the reader. Therefore, the operations $\{\Delta'_n\}_{n\geq 2}$ defined on P above satisfy all A_∞-coalgebra relations on cellular chains of P. $\square$

5. Application to closed compact surfaces

The celebrated classification of closed compact surfaces (cf. [3], for example) states that a closed compact surface of genus g, denoted by X_g, is homeomorphic to a sphere with $g \geq 0$ handles when orientable or a connected sum of $g \geq 1$ real projective planes when unorientable.

To obtain the connected sum $X\#Y$ of two surfaces X and Y, remove the interior of a disk from X and from Y then glue the two surfaces together along their boundaries. Of course, a sphere with $g \geq 1$ handles is the connected sum of g tori, and a Klein bottle is the connected sum of two real projective planes.

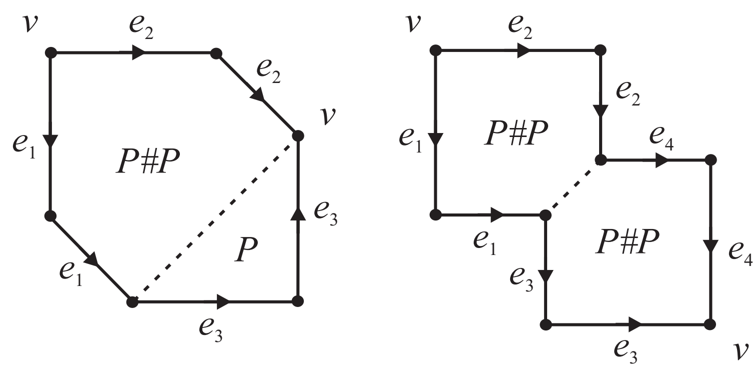

Figure 5. A polygonal decomposition of the connected sum of three and four real projective planes.

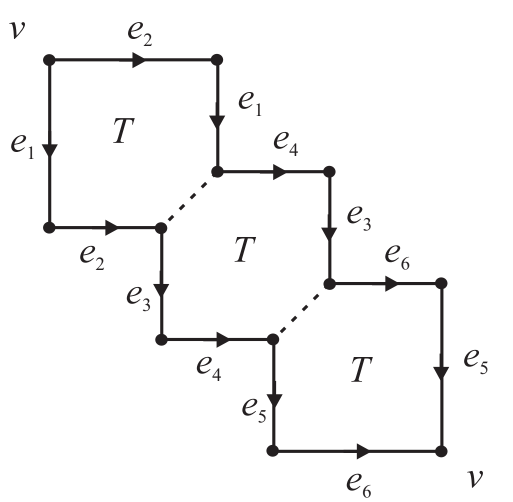

Figure 6. A polygonal decomposition of the connected sum of three tori.

Furthermore, when $g \geq 1$, X_g can be expressed as the quotient of a $4g$-gon when orientable or a $2g$-gon when unorientable as pictured in Figures 5 and 6 (the dotted lines represent common boundaries in the connected sums). To recover X_g from a polygonal decomposition P_g, glue the edges with the same label together so that the arrows directing the edges align. This gluing operation defines the projection $p : P_g \to X_g$, which encodes the topology of X_g.

The cutting and pasting procedures indicated in Figures 5 and 6 can be continued indefinitely, and the directed edges of a particular polygonal decomposition P_g so obtained define a vertex poset with initial and terminal vertices labeled v (all vertices are identified in X_g). These are exactly the configurations to which Corollary 15 applies.

Indeed, the formal A_∞-coalgebra structure on $C_*(P_g)$ given by Corollary 15 projects to a quotient structure on $C_*(X_g)$ in the obvious way. If X_3 is the connected sum of three real projective planes, for example, using the decomposition in Figure 5 we obtain an A_∞-coalgebra structure on $C_*(X_3)$ with non-trivial operations

$$\begin{aligned}\Delta_2(X_3) &= v\otimes X_3 + X_3\otimes v + e_1\otimes e_1 - e_2\otimes e_2 + e_3\otimes e_3\\ \Delta_4(X_3) &= e_1\otimes e_1\otimes e_3\otimes e_3.\end{aligned}$$

Furthermore, each cellular chain in $C_*(P_g)\otimes\mathbb{Z}_2$ projects to a non-bounding cycle in $C_*(X_g)\otimes\mathbb{Z}_2$ so that $H_*(X_g;\mathbb{Z}_2) = C_*(X_g)\otimes\mathbb{Z}_2$.

For a general unorientable X_g, label the edge of the i^{th} real projective plane e_i, and label the vertex v. Let $\lfloor x\rfloor$ denote the floor of x, and for a given s define the sequences

$$\left\{i_p = 2\left\lfloor \tfrac{p+1}{2}\right\rfloor - 1\right\}_{p=1}^{2s} \text{ and } \left\{j_q = 2\left\lfloor \tfrac{q+1}{2}\right\rfloor\right\}_{q=1}^{2s}.$$

Corollary 16. *Let X_g be a closed compact unorientable surface of genus $g\geq 2$ and let P_g be the polygonal decomposition of X_g indicated in Figure 5. The formal A_∞-coalgebra structure on $C_*(P_g)$ projects to a non-trivial A_∞-coalgebra structure on $C_*(X_g)$ with operations $\{\Delta_k\}_{k\geq 2}$ determined by the quotient map $q : P_g \to X_g$ and defined by*

$$\begin{aligned}&\Delta_2(v) = v\otimes v\\ &\Delta_2(e_i) = v\otimes e_i + e_i\otimes v\\ &\Delta_2(X_g) = v\otimes X_g + X_g\otimes v + \sum_{\substack{i=1\\ g\in\{2s-1,2s\}}}^{s} e_{2i-1}\otimes e_{2i-1} - \sum_{\substack{j=1\\ g\in\{2s,2s+1\}}}^{s} e_{2j}\otimes e_{2j}\\ &\Delta_k(X_g) = \sum_{\substack{0<p_1<\cdots<p_k\leq 2s\\ g\in\{2s-1,2s\}}} e_{i_{p_1}}\otimes\cdots\otimes e_{i_{p_k}} - \sum_{\substack{0<q_1<\cdots<q_k\leq 2s\\ g\in\{2s,2s+1\}}} e_{j_{q_1}}\otimes\cdots\otimes e_{j_{q_k}},\ k\geq 3\\ &\Delta_k(\sigma) = 0,\ \textit{for all } \sigma\neq X_g \textit{ and } k\geq 3.\end{aligned}$$

For a general orientable X_g, label the edges along one edge-path from initial to terminal vertex $e_1, e_2, \ldots, e_{2g}$. Let $\hat{e}_{i_{2k-1}} = e_{i_{2k}}$ and $\hat{e}_{i_{2k}} = e_{i_{2k}-1}$, and label the edges along the other edge-path $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{2g}$.

Corollary 17. *Let X_g be a closed compact orientable surface of genus $g\geq 2$ and let P_g be the polygonal decomposition of X_g indicated in Figure 6. The formal A_∞-coalgebra structure on $C_*(P_g)$ projects to a non-trivial A_∞-coalgebra structure on $C_*(X_g)$ with operations $\{\Delta_k\}_{k\geq 2}$ determined by the the quotient map $q : P_g \to X_g$ and defined by*

$$\Delta_2(v) = v\otimes v$$

$$\Delta_2(e_i) = v \otimes e_i + e_i \otimes v$$

$$\Delta_2(X_g) = v \otimes X_g + X_g \otimes v + \sum_{i=1}^{g} e_{2i-1} \otimes e_{2i} - e_{2i} \otimes e_{2i-1}$$

$$\Delta_k(X_g) = \sum_{0<i_1<\cdots<i_k\leq 2g} e_{i_1} \otimes \cdots \otimes e_{i_k} - \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_k},\ k \geq 3$$

$$\Delta_k(\sigma) = 0,\ \textit{for all } \sigma \neq X_g \textit{ and } k \geq 3.$$

It is interesting to note that our definitions of Δ_2 in Corollaries 16 and 17 allow us to read off the cup product on $H_*(X_g;\mathbb{Z}_2)$ directly from the components of Δ_2 without performing additional calculations. In general, one has a choice: Compute cup products using a standard diagonal at the expense of long calculations or construct an application-specific diagonal that minimizes the calculations at the expense of the accompanying combinatorial difficulties.

Finally, when X_g is the sphere S^2, the A_∞-coalgebra structure on $C_*(X_g)$ is clearly degenerate; if X_g is a real projective plane, a torus, or a Klein bottle, there is one non-vanishing A_∞-coalgebra operation on $C_*(X_g)$, namely Δ_2, which induces the non-trivial cup product in cohomology (the formula for Δ_2 on a real projective plane requires independent verification; the proof is straight-forward and omitted). But if X_g is a closed compact surface that is orientable with $g \geq 2$ or unorientable with $g \geq 3$, the combinatorial homotopy coassociative diagonal on $C_*(P_g)$ induces a formal A_∞-coalgebra structure that projects to a strictly coassociative A_∞-coalgebra structure on $C_*(X_g)$ with non-trivial higher order structure determined by the quotient map q. But whether or not the A_∞-coalgebra structure $(H = H_*(X_g;\mathbb{Z}_2), \Delta_k)$ observed here is topologically invariant is an open question.

References

[1] R. Gonzalez-Diaz, J. Lamar, R. Umble. Computing Cup Product in $\mathbb{Z}_2$-cohomology of 3D polyhedral complex. Found. Comp. Math. 14, Springer 2014.

[2] D. Kravatz. Diagonal Approximations on an n-gon and the cohomology Ring of Closed Compact Orientable Surfaces. Senior Honors Thesis. Millersville University Department of Mathematics, 2008. http://sites.millersville.edu/rumble/StudentProjects/Kravatz/finaldraft.pdf

[3] W.S. Massey, Algebraic Topology An Introduction. Harcourt, Brace, and World, Inc., 1967.

[4] Q. Minnich. An A_∞-coalgebra Structure on the Cellular Chains of a Polygon. Senior Honors Thesis. Millersville University Department of Mathematics, 2017. http://sites.millersville.edu/rumble/StudentProjects/Minnich/thesis.pdf

[5] S. Saneblidze and R. Umble, Diagonals on the Permutahedra, Multiplihedra and Associahedra, "J. Homology, Homotopy and Appl., 6 (1) (2004), 363-411.

[6] J. D. Stasheff, Homotopy Associativity of H-spaces I, II, Trans. A. M. S. 108 (1963), 275-312.

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