The Coherent Framed Join and Biassociahedra

Joint work with Samson Saneblidze

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Celebrating the legacies of Jim Stasheff and Murray Gerstenhaber

5 March 2018

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- Dimension $\left|A_1|\cdots|A_r\right| := \left|\pi(A_1|\cdots|A_r)\right|$

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- $\left(\frac{\beta_{ij}}{\alpha_{ij}}\right)^{q \times p}$ is a **bipartition matrix over** $\{\mathbf{a}_i, \mathbf{b}_j\}$ **w.r.t.** R

Bipartition Matrices

- **Example** $\begin{pmatrix} \frac{4|5}{1|0} & \frac{5|4}{3|2} \\ \frac{7|0|6}{0|1|0} & \frac{67}{23} \end{pmatrix}$ is a bipartition matrix

  over $\mathbf{a}_1 = \{1\}$, $\mathbf{a}_2 = \{2,3\}$, $\mathbf{b}_1 = \{4,5\}$, $\mathbf{b}_2 = \{6,7\}$

  with respect to $\begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$

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- **Example:** $\mu_{\{2,3,5\}} 2|1|0|5|4|3 = 12|0|45|3$
- **Extreme cases**:

$$\begin{aligned} \mu_\varnothing (A_1 | \cdots | A_{n+1}) &= A \\ \mu_{\{1,2,\ldots,n\}} (A_1 | \cdots | A_{n+1}) &= A_1 | \cdots | A_{n+1} \end{aligned}$$

Proposition 1

Given a $q \times p$ *bipartition matrix* $\left(\frac{\beta_{ij}}{\alpha_{ij}}\right)$ *over* $\{\mathbf{a}_j, \mathbf{b}_i\}$ *w.r.t.* (r_{ij}), *there is a unique* $q \times p$ *matrix of ordered sets* (λ_{ij}) *such that*

1. $\mu_{\lambda_{1j}}(\alpha_{1j}) = \cdots = \mu_{\lambda_{qj}}(\alpha_{qj})$ *for each* j

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Example:

$$\left(\begin{array}{cc} \frac{45|0}{1|0} & \frac{5|4|0}{0|2|3} \\ \frac{7|0|0|6}{0|1|0|0} & \frac{0|7|6}{2|0|3} \end{array}\right) \overset{\mu_\lambda}{\rightsquigarrow} \left(\begin{array}{cc} \frac{45|0}{1|0} & \frac{45|0}{2|3} \\ \frac{7|6}{1|0} & \frac{7|6}{2|3} \end{array}\right),$$

$$\text{where } \lambda = \left(\begin{array}{cc} \{1\} & \{2\} \\ \{2\} & \{2\} \end{array}\right)$$

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- **Theorem** *A bipartition matrix has a unique indecomposable factorization*

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- $\mathcal{ACP}_{\{1,2,\ldots,9\}} \{2,5,6,8\} = 0|2|0|56|8|0$

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$$C_k = \begin{pmatrix} \frac{\mathbf{b}_{k,1}}{\mathbf{a}_{k,1}} & \cdots & \frac{\mathbf{b}_{k,1}}{\mathbf{a}_{k,s_k}} \\ \vdots & & \vdots \\ \frac{\mathbf{b}_{k,t_k}}{\mathbf{a}_{k,1}} & \cdots & \frac{\mathbf{b}_{k,t_k}}{\mathbf{a}_{k,s_k}} \end{pmatrix}$$

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$$1 = \mathcal{ACP}_1 1 \qquad 56|0|0 = \mathcal{ACP}_{5678} 56$$
$$0|23 = \mathcal{ACP}_{123} 23 \qquad 7|0 = \mathcal{ACP}_{78} 7$$
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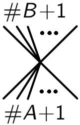
- $$\frac{56|7|8}{1|23|4} = \begin{pmatrix} \frac{56}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{7}{0} & \frac{7}{23} \\ \frac{0}{0} & \frac{0}{23} \end{pmatrix} \begin{pmatrix} \frac{8}{0} & \frac{8}{0} & \frac{8}{0} & \frac{8}{4} \end{pmatrix}$$

Graphical Representation

- $\dfrac{B}{A} \leftarrow$ 

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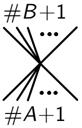
- $\dfrac{B}{A} \leftarrow$ 

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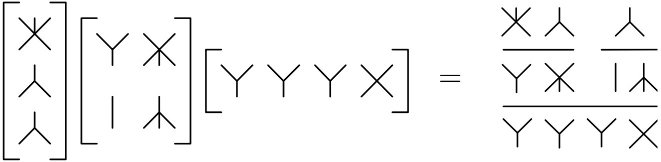
$=$ 

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- **Example** Discard the 1-dim'l indecomposable matrix

$$C = \begin{pmatrix} \frac{0|1}{1|0} & \frac{0|1}{1|0} & \frac{1}{1} \end{pmatrix}$$

Inserting empty blocks in the third entry transforms C into the 3-dim'l decomposable

$$\begin{pmatrix} \frac{0|1}{1|0} & \frac{0|1}{1|0} & \frac{0|1}{0|1} \end{pmatrix} = \begin{pmatrix} \frac{0}{1} & \frac{0}{1} & \frac{0}{0} \\ \frac{0}{1} & \frac{0}{1} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{1} \end{pmatrix}.$$

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- **Example** Preserve all empty blocks in

$$C = \begin{pmatrix} \frac{0}{1} & \frac{0}{3} \\ \frac{0|0}{1|0} & \frac{0|0}{0|3} \end{pmatrix}$$

Removing empty blocks in the second row increases dimension

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  - **coherent** *if column and row coherent*

Coherent Framed Elements

- Given $\mathbf{a}(m)$ and $\mathbf{b}(n)$ of orders m and n, and $r \geq 1$, let

$$\frac{\beta}{\alpha} \in P'_r(\mathbf{a}(m)) \times P'_r(\mathbf{b}(n))$$

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- Otherwise, assume inductively that the set of coherent framed elements $\alpha' \Cup_c \beta'$ has been defined for all $\frac{\beta'}{\alpha'} \in P'(\mathbf{a}(s)) \times P'(\mathbf{b}(t))$ such that $(s, t) \leq (m, n)$ and $s + t < m + n$

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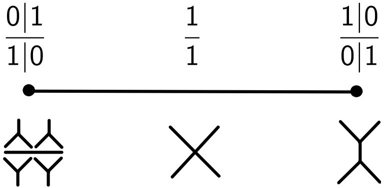

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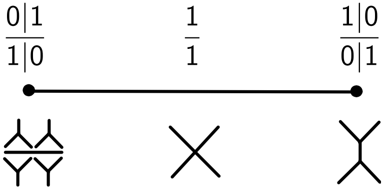

- The Hopf relation holds up to homotopy

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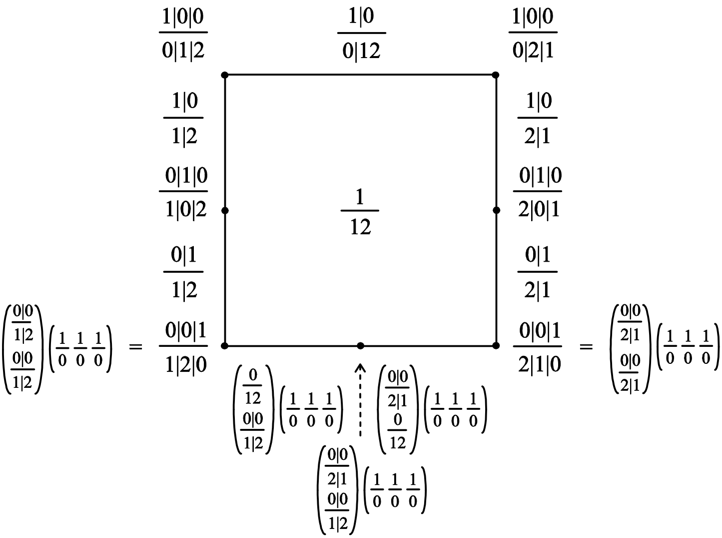

1|0|0 / 0|1|2
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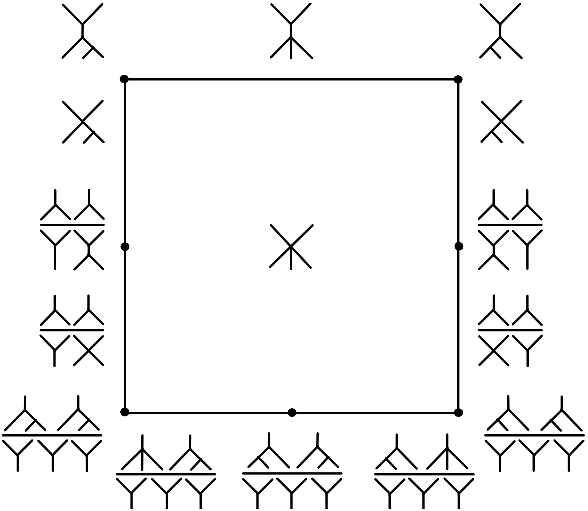

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  - $C = (c_{ij}) \sim C' = \left(c'_{ij}\right)$ *iff* c_{ij} *and* c'_{ij} *differ only in the number or placement of empty blocks* $\frac{0}{0}$
- **Example** $\begin{pmatrix} \frac{0}{1} & \frac{0}{3} \end{pmatrix} = \begin{pmatrix} \frac{0|0}{1|0} & \frac{0|0}{0|3} \end{pmatrix} = \begin{pmatrix} \frac{0|0}{0|1} & \frac{0|0}{3|0} \end{pmatrix}$
- **Definition** *The* **reduced coherent framed join of** $\mathbf{a}(m)$ **and** $\mathbf{b}(n)$ *is the set*

$$\mathbf{a}(m) \circledast_{kk} \mathbf{b}(n) = \mathbf{a}(m) \circledast_{pp} \mathbf{b}(n) / \sim$$

- In $\mathbf{a}(m) \circledast_{kk} \mathbf{b}(n)$
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- In $\mathbf{a}(m) \circledast_{kk} \mathbf{b}(n)$
  - *Dim of a matrix is the sum of the dim's of its entries*
  - *Differential acts on matrix as a derivation of its entries*

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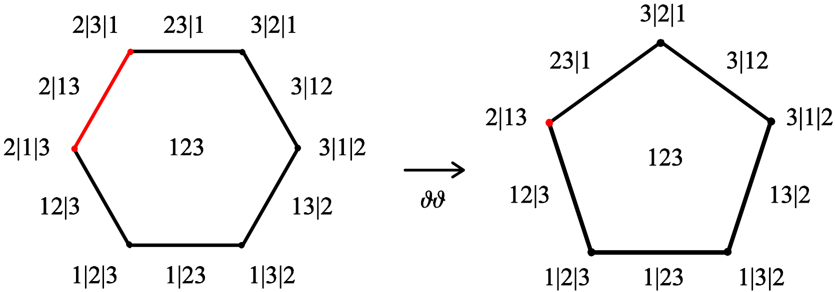
- $KK_{n+1,m+1} \leftrightarrow \mathfrak{m} \circledast_{kk} \mathfrak{n}$
- In $KK_{1,4} \leftrightarrow \mathfrak{3} \circledast_{kk} \mathfrak{o}$ we have

$$\left(\frac{0}{2}\right)\left(\frac{0}{1}\ \ \frac{0}{3}\right) = \left(\frac{0}{2}\right)\left(\frac{0|0}{1|0}\ \ \frac{0|0}{0|3}\right) = \left(\frac{0}{2}\right)\left(\frac{0|0}{0|1}\ \ \frac{0|0}{3|0}\right)$$

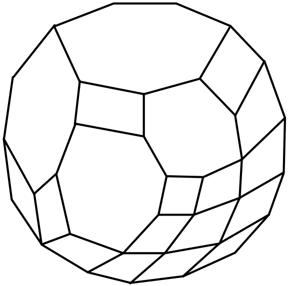
so that

$$\frac{0|0}{2|13} = \frac{0|0|0}{2|1|3} = \frac{0|0|0}{2|3|1}$$

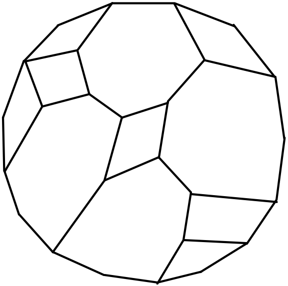
Stasheff's Associahedron K(4)



KK(3,3)



Front view



Rear view

- $\partial KK_{3,3}$ consists of 8 heptagons and 22 squares

A-infinity Bialgebras

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- **Theorem** *A non-trivial A_∞-coalgebra structure on $H_*(X;\mathbb{Q})$ induces a non-trivial A_∞-bialgebra structure on $H_*(\Omega\Sigma X;\mathbb{Q})$*
- The A_∞-bialgebra structure on $H_*(\Omega\Sigma X;\mathbb{Q})$ is a rational homology invariant
- Prior to this work, all known rational homology invariants of $\Omega\Sigma X$ were trivial

Happy Birthday
Jim and Murray!