Tensor Products of $A_\infty$-algebras with Homotopy Inner Products

(Joint work with Thomas Tradler, CUNY)

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Tensor Products of $A$-infinity Algebras

- Let $K = \square K_n$ denote Stasheff’s associahedra
Tensor Products of $A$-infinity Algebras

- Let $K = \biguplus K_n$ denote Stasheff’s associahedra

- Let $R$ be a commutative ring with unity
Tensor Products of $A$-infinity Algebras

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- Let $R$ be a commutative ring with unity
- Let $(A, \{\mu_n\})$ and $(B, \{\nu_n\})$ be $A_\infty$-algebras over $R$
Tensor Products of $A$-infinity Algebras

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- Let $(A, \{\mu_n\})$ and $(B, \{\nu_n\})$ be $A_\infty$-algebras over $R$
- A diagonal on cellular chains

$$Δ_K : C_*K \rightarrow C_*K \otimes C_*K$$

was constructed by Saneblidze-U and Markl-Snider
Tensor Products of $A$-infinity Algebras

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- Let $R$ be a commutative ring with unity
- Let $(A, \{\mu_n\})$ and $(B, \{\nu_n\})$ be $A_\infty$-algebras over $R$
- A diagonal on cellular chains $\Delta_K : C_* K \rightarrow C_* K \otimes C_* K$
  was constructed by Saneblidze-U and Markl-Snider
- $\Delta_K$ induces an $A_\infty$-algebra structure $\{\varphi_n\}$ on $A \otimes B$
Define $\varphi_1 = \mu_1 \otimes 1 + 1 \otimes \nu_1$
Tensor Products of $A$-infinity Algebras

- Define $\varphi_1 = \mu_1 \otimes 1 + 1 \otimes \nu_1$

- Given operadic representations of $A_\infty$-structures

\[
\{ \zeta_n : C_* K \to \text{Hom} (A^\otimes_n, A) \}_{n \geq 2}
\]

\[
\{ \xi_n : C_* K \to \text{Hom} (B^\otimes_n, B) \}_{n \geq 2}
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Tensor Products of $A$-infinity Algebras

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- Given operadic representations of $A_\infty$-structures

  \[ \{ \zeta_n : C_* K \to \text{Hom} \left( A \otimes^n, A \right) \}_{n \geq 2} \]

  \[ \{ \xi_n : C_* K \to \text{Hom} \left( B \otimes^n, B \right) \}_{n \geq 2} \]

- Define a representation $\theta_n$ by the composition

\[
\begin{array}{ccc}
C_* K & \xrightarrow{\theta_n} & \text{Hom} \left( (A \otimes B) \otimes^n, A \otimes B \right) \\
\Delta_K & \downarrow & \uparrow \approx \\
C_* K \otimes C_* K & \xrightarrow{\zeta_n \otimes \xi_n} & \text{Hom} \left( A \otimes^n, A \right) \otimes \text{Hom} \left( B \otimes^n, B \right)
\end{array}
\]
Then $\theta_n$ sends the top-dimensional cell $e^{n-2} \subset K_n$ to

$$\varphi_n = \left[ (\zeta_n \otimes \xi_n) \Delta_K (e^{n-2}) \right] \sigma_n$$

where $\sigma_n : A^\otimes n \otimes B^\otimes n \to (A \otimes B)^\otimes n$ is canonical permutation
Tensor Products of $A$-infinity Algebras

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- And lower-dimensional faces to $\circ_i$-compositions
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And lower-dimensional faces to \( \circ_i \)-compositions

\[
\varphi_2 = (\mu_2 \otimes \nu_2) \sigma_2
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And lower-dimensional faces to $\circ_i$-compositions

$$\varphi_2 = (\mu_2 \otimes \nu_2) \sigma_2$$

$$\varphi_3 = \mu_2 (\mu_2 \otimes 1) \otimes \nu_3 + \mu_3 \otimes \nu_2 (1 \otimes \nu_2)$$
Tensor Products of $A$-infinity Algebras

- Then $\theta_n$ sends the top-dimensional cell $e_{n-2} \subset K_n$ to
  \[
  \varphi_n = \left[ (\zeta_n \otimes \zeta_n) \Delta_K (e^{n-2}) \right] \sigma_n
  \]
  where $\sigma_n : A^{\otimes n} \otimes B^{\otimes n} \rightarrow (A \otimes B)^{\otimes n}$ is canonical permutation

- And lower-dimensional faces to $\circ_i$-compositions

- $\varphi_2 = (\mu_2 \otimes \nu_2) \sigma_2$

- $\varphi_3 = \mu_2 (\mu_2 \otimes 1) \otimes \nu_3 + \mu_3 \otimes \nu_2 (1 \otimes \nu_2)$

- And so on...
An $A\infty$-algebra $(A, \{\mu_n\})$ is **cyclic** if

$A$ is equipped with a cyclically invariant inner product

$$\langle \mu_n(a_1, \ldots, a_n), a_{n+1} \rangle = \langle \mu_n(a_2, \ldots, a_{n+1}), a_1 \rangle$$
Cyclic $A$-infinity Algebras

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- What is the structure of $A \otimes B$ when $A$ and $B$ are cyclic?
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What is the structure of $A \otimes B$ when $A$ and $B$ are cyclic?

Inner products $\langle - , - \rangle_A$ and $\langle - , - \rangle_B$ induce an inner product

$$\langle a | b, c | d \rangle_{A \otimes B} = \langle a, c \rangle_A \langle b, d \rangle_B$$
The differential $\varphi_1$ is cyclically invariant (ignoring signs):

$$\langle \varphi_1 (a|b) , c|d \rangle = \langle \mu_1 (a) | b + a| \nu_1 (b) , c|d \rangle$$
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\[
= \langle \mu_1 (a) , c \rangle_A \langle b , d \rangle_B + \langle a , c \rangle_A \langle \nu_1 (b) , d \rangle_B
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The differential $\varphi_1$ is cyclically invariant (ignoring signs):

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\langle \varphi_1 (a|b), c|d \rangle = \langle \mu_1 (a) | b + a | \nu_1 (b) , c|d \rangle \\
= \langle \mu_1 (a) , c \rangle_A \langle b, d \rangle_B + \langle a, c \rangle_A \langle \nu_1 (b) , d \rangle_B \\
= \langle \mu_1 (c) , a \rangle_A \langle d, b \rangle_B + \langle c, a \rangle_A \langle \nu_1 (d) , b \rangle_B
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Tensor Product of Cyclic $A$-infinity Algebras

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$$= \langle \mu_1 (c) | d + c|\nu_1 (d) , a|b \rangle$$

$$= \langle \varphi_1 (c|d) , a|b \rangle$$
The product $\varphi_2$ is cyclically invariant:

$$\langle \varphi_2 (a|b, c|d) , e|f \rangle = \langle \mu_2 (a, c) |v_2 (b, d) , e|f \rangle$$
The product $\varphi_2$ is cyclically invariant:

$$\langle \varphi_2 (a \vert b, c \vert d), e \vert f \rangle = \langle \mu_2 (a, c) \vert \nu_2 (b, d), e \vert f \rangle$$

$$= \langle \mu_2 (a, c), e \rangle_A \langle \nu_2 (b, d), f \rangle_B$$
Tensor Product of Cyclic $A$-infinity Algebras

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\langle \varphi_2 (a|b,c|d), e|f \rangle = \langle \mu_2 (a,c) | \nu_2 (b,d), e|f \rangle \\
= \langle \mu_2 (a,c), e \rangle_A \langle \nu_2 (b,d), f \rangle_B \\
= \langle \mu_2 (c,e), a \rangle_A \langle \nu_2 (d,f), b \rangle_B
$$
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Tensor Product of Cyclic $A$-infinity Algebras

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$$

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= \langle \mu_2 (c, e) | \nu_2 (d, f) , a|b \rangle
$$

$$
= \langle \varphi_2 (c|d , e|f) , a|b \rangle
$$
Tensor Product of Cyclic $A$-infinity Algebras

- But $\varphi_3$ is not cyclically invariant because...
Tensor Product of Cyclic $A$-infinity Algebras

- But $\varphi_3$ is not cyclically invariant because...

- $\langle \varphi_3 (a|b, c|d, e|f), g|h \rangle = \langle \varphi_3 (c|d, e|f, g|h), a|b \rangle$ implies

  (1) $\langle \mu_2 (\mu_2 (a, c), e), g \rangle_A = \langle \mu_2 (\mu_2 (c, e), g), a \rangle_A$

  (2) $\langle \nu_2 (b, \nu_2 (d, f)), h \rangle_B = \langle \nu_2 (d, \nu_2 (f, h)), b \rangle_B$
But $\varphi_3$ is not cyclically invariant because...

\[ \langle \varphi_3 (ab, cd, ef), gh \rangle = \langle \varphi_3 (cd, ef, gh), ab \rangle \] implies

\begin{align*}
(1) & \quad \langle \mu_2 (\mu_2 (a, c), e), gh \rangle_A = \langle \mu_2 (\mu_2 (c, e), g), a \rangle_A \\
(2) & \quad \langle v_2 (b, v_2 (d, f)), h \rangle_B = \langle v_2 (d, v_2 (f, h)), b \rangle_B
\end{align*}

Which only hold up to homotopy
Tensor Product of Cyclic A-infinity Algebras

- Cyclicity and homotopy associativity give chain homotopies
Tensor Product of Cyclic $A$-infinity Algebras

- Cyclicity and homotopy associativity give chain homotopies

- For relation (1):

\[
\langle \mu_2 (\mu_2 (a, c), e), g \rangle = \langle [\mu_1, \mu_3] (a, c, e) \pm \mu_2 (a, \mu_2 (c, e)), g \rangle
\]

\[
= \langle [\mu_1, \mu_3] (a, c, e), g \rangle \pm \langle \mu_2 (\mu_2 (c, e), g), a \rangle
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Tensor Product of Cyclic $A$-infinity Algebras

- Cyclicity and homotopy associativity give chain homotopies

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$$= \langle [\mu_1, \mu_3] (a, c, e), g \rangle \pm \langle \mu_2 (\mu_2 (c, e), g), a \rangle$$

- Another application of cyclicity gives the chain homotopy

$$(\langle \mu_3, - \rangle \circ d) (a, c, e, g) =$$

$$\langle \mu_2 (\mu_2 (a, c), e), g \rangle \pm \langle \mu_2 (\mu_2 (c, e), g), a \rangle$$

where $d$ is the linear extension of $\mu_1$
Tensor Product of Cyclic $A$-infinity Algebras

- Chain homotopies (1) and (2) induce a chain homotopy

$$\varrho_{2,0} : (A \otimes B)^{\otimes 4} \to R$$

such that

$$(\varrho_{2,0} \circ d) (a|b, c|d, e|f, g|h) =$$

$$\langle \varphi_3 (a|b, c|d, e|f), g|h \rangle - \langle \varphi_3 (c|d, e|f, g|h), a|b \rangle$$
Tensor Product of Cyclic $A$-infinity Algebras

- Chain homotopies (1) and (2) induce a chain homotopy
  \[ \varrho_{2,0} : (A \otimes B)^{\otimes 4} \to R \text{ such that} \]
  \[ (\varrho_{2,0} \circ d) (a \vert b, c \vert d, e \vert f, g \vert h) = \]
  \[ \langle \varphi_3 (a \vert b, c \vert d, e \vert f) , g \vert h \rangle - \langle \varphi_3 (c \vert d, e \vert f, g \vert h) , a \vert b \rangle \]

- $\varrho_{2,0}$ extends to an infinite family of higher homotopies $\{ \varrho_{k,l} \}$
Tensor Product of Cyclic $A$-infinity Algebras

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(\varrho_{2,0} \circ d) (a|b, c|d, e|f, g|h) = 
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$$

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- **Conclusion:** The tensor product of cyclic $A_\infty$-algebras is cyclic up to homotopy, and in fact...
Tensor Product of Cyclic $A$-infinity Algebras

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- **Conclusion:** The tensor product of cyclic $A_\infty$-algebras is cyclic up to homotopy, and in fact...

- There exists additional bimodule structure s.t. $\left( A \otimes B, \{ \varphi_n \}, \{ \varrho_{k,l} \} \right)$ is an $A_\infty$-algebra with homotopy inner products (HIPs)
A-infinity Algebras with HIPs

- **Goal:** Define tensor product of general $A_\infty$-algebras with HIPs
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- **Goal**: Define tensor product of *general* $A_{\infty}$-algebras with HIPs

- An $A_{\infty}$-*algebra with homotopy inner products* consists of
A-infinity Algebras with HIPs

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- An $A_\infty$-algebra with homotopy inner products consists of
  
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A-infinity Algebras with HIPs

- **Goal:** Define tensor product of *general* $A_\infty$-algebras with HIPs

- An $A_\infty$-algebra with homotopy inner products consists of

  1. an $A_\infty$-algebra $(A, \{\mu_n\})$

  2. a compatible family of higher inner products

$$\left\{ q_{j,k} : A \otimes A^{\otimes j} \otimes A \otimes A^{\otimes k} \to R \right\}$$
A-infinity Algebras with HIPs

- **Goal:** Define tensor product of *general* $A_\infty$-algebras with HIPs

- An $A_\infty$-algebra with homotopy inner products consists of

1. an $A_\infty$-algebra $(A, \{\mu_n\})$

2. a compatible family of higher inner products

\[ \left\{ \varrho_{j,k} : A \otimes A^{\otimes j} \otimes A \otimes A^{\otimes k} \to R \right\} \]

3. a compatible family of module maps

\[ \left\{ \lambda_{j,k} : A^{\otimes j} \otimes A \otimes A^{\otimes k} \to A \right\} \]
A-infinity Algebras with HIPs

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- Structure relations encoded by a 3-colored operad $C_\ast A$
A-infinity Algebras with HIPs

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- Structure relations encoded by a 3-colored operad $C_\ast A$

- Identified with cellular chains of contractible pairahedra
The 3-colored operad $CA$

$C_* \mathcal{A}$ is generated by three types of planar diagrams

**Colors of leaves and root:** Empty, thin, thick

1. **Planar trees:** Control $A_\infty$-algebra structure
   - *Thin leaves and root*
The 3-colored operad CA

2. **Module trees**: Control homotopy bimodule structure
   - *Thick vertical root and leaf*
   - *j thin leaves in left half-plane*
   - *k thin leaves in right half-plane*
The 3-colored operad $CA$

3. **Inner product diagrams**: Control HIP structure
   - *Empty root and two thick horizontal leaves*
   - *$j$ thin leaves in upper half-plane*
   - *$k$ thin leaves in lower half-plane*
Operadic Structure of CA

- Compose planar trees in the usual way

- Compose a module diagram $M$ with a planar tree $T$ by attaching the root of $T$ to a thin leaf of $M$.

- Compose module trees by attaching thick root of 2nd to thick leaf of 1st.

- Compose an IP diagram with a module tree $M$ by attaching thick root of $M$ to a thick leaf of $I$.

- Two inner product diagrams cannot be composed.
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Operadic Structure of $CA$

- Compose planar trees in the usual way
- Compose a module diagram $M$ with a planar tree $T$ by attaching the root of $T$ to a thin leaf of $M$
- Compose module trees by attaching thick root of $2^{nd}$ to thick leaf of $1^{st}$

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DG Module Structure of \( CA \)

- Let \( D \) be a diagram – a generator of \( C^*A \)
DG Module Structure of $CA$

- Let $D$ be a diagram – a generator of $C_* A$
- $\mathcal{L}(D) = \{\text{Leaves of } D\}$
DG Module Structure of $CA$

- Let $D$ be a diagram – a generator of $C \ast A$
- $\mathcal{L}(D) = \{\text{Leaves of } D\}$
- $\mathcal{E}(D) = \{(\text{Internal}) \text{ edges of } D\}$
DG Module Structure of CA

- Let $D$ be a diagram – a generator of $C\star A$

- $\mathcal{L}(D) = \{\text{Leaves of } D\}$

- $\mathcal{E}(D) = \{\text{(Internal) edges of } D\}$

- **Degree:** $|D| := \#\mathcal{L}(D) - \#\mathcal{E}(D) - 2$
DG Module Structure of $CA$

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- **Boundary:** $\partial_C(D) := \sum_{D'/e=D} D'$, where $e$ is an edge of $D'$
DG Module Structure of CA

- Let $D$ be a diagram – a generator of $\mathcal{C}_*\mathcal{A}$

- $\mathcal{L}(D) = \{\text{Leaves of } D\}$

- $\mathcal{E}(D) = \{\text{(Internal) edges of } D\}$

- **Degree:** $|D| := \#\mathcal{L}(D) - \#\mathcal{E}(D) - 2$

- **Boundary:** $\partial_C(D) := \sum_{D'/e=D} D'$, where $e$ is an edge of $D'$

- $\partial_C(D)$ is the sum of all diagrams obtained from $D$ by inserting a single edge
Coloring in $CA$

- $0 = \text{empty}; \ 1 = \text{thin}; \ 2 = \text{thick}$
0 = empty; 1 = thin; 2 = thick

The **coloring** of a diagram $D$ with $n$ leaves is a pair

$$x \times y = (x_1, \ldots, x_n) \times y \in \mathbb{Z}_3^{n+1}$$

- $x_i$ is the color of leaf $i$
- $y$ is the color of the root
Coloring in \( CA \)

- \( 0 = \) empty; \( 1 = \) thin; \( 2 = \) thick

- The **coloring** of a diagram \( D \) with \( n \) leaves is a pair

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  - \( x_i \) is the color of leaf \( i \)
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- \( C_x A_y \) is generated by diagrams of coloring \( x \times y \)
Coloring in CA

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  - $x_i$ is the color of leaf $i$
  - $y$ is the color of the root

- $C_\star A_x^y$ is generated by diagrams of coloring $x \times y$

- **Example**: $C_\star A_1^{11\ldots1}$ is generated by planar trees
Example

\( C_* \mathbb{A}_0^{2112} \) generated by IP diagrams \( \leftrightarrow \) faces of pairahedron \( l_{2,0} \):
Following the $W$-construction of Boardman and Vogt, there is a cubical subdivision $Q\mathcal{A}$ of $C\mathcal{A}$ s.t.
Cubical Subdivision of $CA$

- Following the $W$-construction of Boardman and Vogt, there is a cubical subdivision $Q_A$ of $C_A$ s.t.

- $Q_A$ is a 3-colored operad
Following the $W$-construction of Boardman and Vogt, there is a cubical subdivision $Q\mathcal{A}$ of $C\mathcal{A}$ s.t.

- $Q\mathcal{A}$ is a 3-colored operad
- $Q\mathcal{A}$ is generated by all metric diagrams $(D, g)$, where
  - $D$ is a generator of $C\mathcal{A}$
  - $g : \mathcal{E}(D) \to \{m, n\}$ labels the (internal) edges of $D$
    either “$m$” (metric) or “$n$” (non-metric)
Example

Cubical subdivision of $l_{2,0}$ (only metric labels are displayed):
The 3-Colored Operad QA

- When composing diagrams: Label the new edge "n"
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- Degree: \(|(D, g)| := \# \text{ metric edges}\)
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Boundary: \(\partial_Q(D) := \sum_{\text{metric edges } e} D/e + D_e\)

where \(D_e\) is obtained from \(D\) by relabeling \(e\) non-metric
The 3-Colored Operad QA

- **When composing diagrams:** Label the new edge “n”

- **Degree:** \(|(D, g)| := \# \text{ metric edges}\)

- **Boundary:** 

  \[
  \partial_Q(D) := \sum_{\text{metric edges } e} D/e + D_e
  \]

  where \(D_e\) is obtained from \(D\) by relabeling \(e\) non-metric

- \[
  \partial_Q\left(\begin{array}{c}
  \text{\textbullet} \\
  m & m
  \end{array}\right) = \begin{array}{c}
  \text{\textbullet} \\
  m
  \end{array} + \begin{array}{c}
  \text{\textbullet} \\
  m & n
  \end{array} + \begin{array}{c}
  \text{\textbullet} \\
  m
  \end{array} + \begin{array}{c}
  \text{\textbullet} \\
  n & m
  \end{array}
  \]
Example - Boundary of a Metric Square
The Homotopy Equivalence $q : CA \rightarrow QA$

- Let $m$ denote the constant map $m(e) = m$
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- $C_0A$ is generated by *binary diagrams* ($\#\mathcal{L} = \#\mathcal{E} + 2$)
The Homotopy Equivalence $q : CA \rightarrow QA$

- Let $m$ denote the constant map $m(e) = m$

- $C_0A$ is generated by binary diagrams ($\#L = \#E + 2$)

- **Definition.** On a corolla $c \in C_*A^x_\gamma$ define

  $$q(c) = \sum_{B \in C_0A^x_\gamma} (B, m)$$
The Homotopy Equivalence $q : CA \longrightarrow QA$

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- **Definition.** On a corolla $c \in C_*\mathcal{A}_y^x$ define

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- A general diagram is a $\circ_i$-composition of corollas
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- A general diagram is a $\circ_i$-composition of corollas

- Extend $q$ to $\circ_i$-compositions multiplicatively:

\[
q(c \circ_i c') = q(c) \circ_i q(c')
\]
The Poset of Binary Diagrams in CA

- Extend Tamari ordering on binary trees to binary diagrams
The Poset of Binary Diagrams

- $\mathcal{B}$ denotes the poset of all binary diagrams
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The Poset of Binary Diagrams

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- $\mathcal{B}_D \subset \mathcal{B}$ is the vertex poset of a diagram $D$
- $\mathcal{B}_D$ has minimal element $D_{\text{min}}$ and maximal element $D_{\text{max}}$
- Example: $\mathcal{B}_{I_{2,0}}$
The 2-Sided Homotopy Inverse $p : QA \rightarrow CA$

- **Definition.** On a *fully metric* $(D, m) \in Q_k \mathbb{A}_y^x$ define

\[ p(D, m) = \sum_{S \in C_k \mathbb{A}_y^x, S_{\text{max}} \leq D_{\text{min}}} S \]
The 2-Sided Homotopy Inverse $p : QA \rightarrow CA$

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\]

- **Proposition**
The 2-Sided Homotopy Inverse $p : QA \rightarrow CA$

- **Definition.** On a fully metric $(D, m) \in Q_kA^x_y$ define
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  p(D, m) = \sum_{S \in C_kA^x_y} S \\
  S_{\text{max}} \leq D_{\text{min}}
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- **Proposition**
  1. On a corolla $c \in Q_\ast A$
      \[
      p(c) = c_{\text{min}} \in B_c
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The 2-Sided Homotopy Inverse $p : QA \rightarrow CA$

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- **Proposition**

1. On a corolla $c \in Q_* A$

$$p(c) = c_{\text{min}} \in B_c$$

2. On a fully metric binary diagram $(B, m)$

$$p(B, m) = \begin{cases} 
  c, & \text{if } B = c_{\text{max}} \text{ for some corolla } c \\
  0, & \text{otherwise}
\end{cases}$$
The 2-Sided Homotopy Inverse $p : QA \rightarrow CA$

- A metric diagram is a $o_i$-composition of fully metric diagrams.
The 2-Sided Homotopy Inverse $p : QA \rightarrow CA$

- A metric diagram is a $\circ_i$-composition of fully metric diagrams
- Extend $p$ to $\circ_i$-compositions multiplicatively

\[ p [(D, m) \circ_i (D', m)] = p (D, m) \circ_i p (D', m) \]
The 2-Sided Homotopy Inverse $p : QA \rightarrow CA$

- A metric diagram is a $\circ_i$-composition of fully metric diagrams
- Extend $p$ to $\circ_i$-compositions multiplicatively
  \[ p \left[ (D, m) \circ_i (D', m) \right] = p(D, m) \circ_i p(D', m) \]
- Theorem
The 2-Sided Homotopy Inverse $p : QA \longrightarrow CA$

- A metric diagram is a $\circ_i$-composition of fully metric diagrams.

- Extend $p$ to $\circ_i$-compositions multiplicatively.

\[ p \left[ (D, m) \circ_i (D', m) \right] = p(D, m) \circ_i p(D', m) \]

- **Theorem**

1. $p$ and $q$ are chain maps
The 2-Sided Homotopy Inverse $p : QA \rightarrow CA$

- A metric diagram is a $\circ_i$-composition of fully metric diagrams
- Extend $p$ to $\circ_i$-compositions multiplicatively
  \[ p \left[ (D, m) \circ_i (D', m) \right] = p (D, m) \circ_i p (D', m) \]
- **Theorem**
  1. $p$ and $q$ are chain maps
  2. $pq = \text{Id}$ and $qp \simeq \text{Id}$
The Diagonal on $\mathcal{Q}A$

- Given $(D, g) \in Q_* \mathcal{A}$, let $X \subseteq \{\text{metric edges of } D\}$
The Diagonal on $QA$

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- Let $\overline{X} = \{\text{metric edges of } D\} - X$
The Diagonal on QA

- Given \((D, g) \in Q_\ast A\), let \(X \subseteq \{\text{metric edges of } D\}\)
- Let \(\overline{X} = \{\text{metric edges of } D\} - X\)
- Obtain \(D/X\) from \(D\) by *contracting the edges of \(X\)*
The Diagonal on $Q_A$

- Given $(D, g) \in Q_A$, let $X \subseteq \{\text{metric edges of } D\}$
- Let $\overline{X} = \{\text{metric edges of } D\} - X$
- Obtain $D/X$ from $D$ by *contracting the edges of $X$*
- Obtain $D_{\overline{X}}$ from $D$ by *relabeling the edges in $\overline{X}$ non-metric*
The Diagonal on QA

- Given \((D, g) \in Q_*A\), let \(X \subseteq \{\text{metric edges of } D\}\)
- Let \(\overline{X} = \{\text{metric edges of } D\} - X\)
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- Obtain \(D_{\overline{X}}\) from \(D\) by \textit{relabeling the edges in } \(\overline{X}\) \textit{non-metric}
- Serre’s diagonal on \(I^n\) induces a coassociative diagonal

\[\Delta_Q : Q_*A \rightarrow Q_*A \otimes Q_*A\]
The Diagonal on $QA$

- Given $(D, g) \in Q_*A$, let $X \subseteq \{\text{metric edges of } D\}$
- Let $\overline{X} = \{\text{metric edges of } D\} - X$
- Obtain $D/X$ from $D$ by contracting the edges of $X$
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- Serre’s diagonal on $I^n$ induces a coassociative diagonal
  \[\Delta_Q : Q_*A \to Q_*A \otimes Q_*A\]
- Given by
  \[\Delta_Q(D) = \sum_X D/X \otimes D_{\overline{X}}\]
The Induced Diagonal on $CA$

- $\Delta_Q$ induces a non-coassociative diagonal on $C_\ast A$

$$
\Delta_C : C_\ast A \xrightarrow{q} Q_\ast A \xrightarrow{\Delta_Q} Q_\ast A \otimes Q_\ast A \xrightarrow{p \otimes p} C_\ast A \otimes C_\ast A
$$
The Induced Diagonal on $CA$

- $\Delta_Q$ induces a non-coassociative diagonal on $C_\ast A$

\[
\Delta_C : C_\ast A \xrightarrow{q} Q_\ast A \xrightarrow{\Delta_Q} Q_\ast A \otimes Q_\ast A \xrightarrow{p \otimes p} C_\ast A \otimes C_\ast A
\]

- On a corolla $c \in C_k A^x_y$:

\[
\Delta_C(c) = \sum_{S \otimes T \in C_i A^x_y \otimes C_{k-i} A^x_y} S \otimes T
\]

\[
S_{\text{max}} \leq T_{\text{min}}
\]
Examples

\[ \Delta_C \left( \begin{array}{c} \longrightarrow \\ \end{array} \right) = \begin{array}{c} \longrightarrow \\ \end{array} \otimes \begin{array}{c} \longrightarrow \\ \end{array} \]

\[ \Delta_C \left( \begin{array}{c} \downarrow \\ \end{array} \right) = \begin{array}{c} \downarrow \\ \end{array} \otimes \begin{array}{c} \downarrow \\ \end{array} + \begin{array}{c} \downarrow \end{array} \otimes \begin{array}{c} \downarrow \end{array} \]

\[ \Delta_C \left( \begin{array}{c} \triangledown \\ \end{array} \right) = \begin{array}{c} \triangledown \otimes \begin{array}{c} \triangledown \\ \end{array} + \begin{array}{c} \triangledown \otimes \begin{array}{c} \triangledown \\ \end{array} \end{array} + \begin{array}{c} \triangledown \end{array} \otimes \begin{array}{c} \triangledown \\ \end{array} \]

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Given representations of $A_\infty$-algebras with HIPs

$$\{ \phi_n : C_\ast \mathcal{A} \to \text{Hom} (A \otimes^n, A) \}_{n \geq 2}$$

$$\{ \psi_n : C_\ast \mathcal{A} \to \text{Hom} (B \otimes^n, B) \}_{n \geq 2}$$
Tensor Products of $A$-infinity Algebras with HIPs

- Given representations of $A_\infty$-algebras with HIPs

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\]

- Define the representation $\epsilon_n$ to be the composition

\[
\begin{array}{ccc}
C_* A & \xrightarrow{\epsilon_n} & \text{Hom} \left( (A \otimes B) \otimes^n, A \otimes B \right) \\
\Delta_C & \downarrow & \uparrow \cong \\
C_* A \otimes C_* A & \xrightarrow{\phi_n \otimes \psi_n} & \text{Hom} \left( A \otimes^n, A \right) \otimes \text{Hom} \left( B \otimes^n, B \right)
\end{array}
\]
Tensor Products of $A$-infinity Algebras with HIPs

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\[ \{ \phi_n : C_*A \to \text{Hom} (A^\otimes n, A) \}_{n \geq 2} \]

\[ \{ \psi_n : C_*A \to \text{Hom} (B^\otimes n, B) \}_{n \geq 2} \]

- Define the representation $\varepsilon_n$ to be the composition

\[
\begin{array}{ccc}
C_*A & \xrightarrow{\varepsilon_n} & \text{Hom} \left( \left( A \otimes B \right)^\otimes n, A \otimes B \right) \\
\Delta_C & \downarrow & \\
C_*A \otimes C_*A & \xrightarrow{\phi_n \otimes \psi_n} & \text{Hom} (A^\otimes n, A) \otimes \text{Hom} (B^\otimes n, B)
\end{array}
\]

- $(A \otimes B, \varphi_1, \varepsilon (C_*A))$ is an $A_\infty$-algebra with HIPs
Given representations of $A_\infty$-algebras with HIPs

$$\left\{ \phi_n : C_* A \to \text{Hom} \left( A^\otimes n, A \right) \right\}_{n \geq 2}$$

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Define the representation $\varepsilon_n$ to be the composition

$$C_* A \xrightarrow{\varepsilon_n} \text{Hom} \left( (A \otimes B)^\otimes n, A \otimes B \right)$$

$$(A \otimes B, \varphi_1, \varepsilon(C_* A)) \text{ is an } A_\infty\text{-algebra with HIPs}$$

Paper to appear in TAMS
Thank you!