Tensor Products of A_∞-algebras with Homotopy Inner Products

(Joint work with Thomas Tradler, CUNY)

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was constructed by Saneblidze-U and Markl-Snider

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  was constructed by Saneblidze-U and Markl-Snider
- Δ_K induces an A_∞-algebra structure $\{\varphi_n\}$ on $A \otimes B$

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- Define a representation θ_n by the composition

$$\begin{array}{ccc} C_* K & \overset{\theta_n}{\dashrightarrow} & Hom\left((A \otimes B)^{\otimes n}, A \otimes B\right) \\ \Delta_K \downarrow & & \uparrow \approx \\ C_* K \otimes C_* K & \xrightarrow{\zeta_n \otimes \xi_n} & Hom\left(A^{\otimes n}, A\right) \otimes Hom\left(B^{\otimes n}, B\right) \end{array}$$

Tensor Products of A-infinity Algebras

- Then θ_n sends the top-dimensional cell $e^{n-2} \subset K_n$ to

$$\varphi_n = \left[(\zeta_n \otimes \xi_n) \, \Delta_K \left(e^{n-2} \right) \right] \sigma_n$$

where $\sigma_n : A^{\otimes n} \otimes B^{\otimes n} \longrightarrow (A \otimes B)^{\otimes n}$ is canonical permutation

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- And so on...

Cyclic A-infinity Algebras

- An A_∞-algebra $(A, \{\mu_n\})$ is **cyclic** if

  A is equipped with a cyclically invariant inner product

$$\langle \mu_n (a_1, \ldots, a_n), a_{n+1} \rangle = \langle \mu_n (a_2, \ldots, a_{n+1}), a_1 \rangle$$

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- What is the structure of $A \otimes B$ when A and B are *cyclic?*

- Inner products $\langle -, - \rangle_A$ and $\langle -, - \rangle_B$ induce an inner product

  $$\langle a|b, c|d \rangle_{A \otimes B} = \langle a, c \rangle_A \langle b, d \rangle_B$$

Tensor Product of Cyclic A-infinity Algebras

- The differential φ_1 is cyclically invariant (ignoring signs):

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- Which only hold up to homotopy

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- For relation (1):

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- Another application of cyclicity gives the chain homotopy

$$(\langle \mu_3, - \rangle \circ d) (a, c, e, g) =$$

$$\langle \mu_2 (\mu_2 (a, c), e), g \rangle \pm \langle \mu_2 (\mu_2 (c, e), g), a \rangle$$

where d is the linear extension of μ_1

Tensor Product of Cyclic A-infinity Algebras

- Chain homotopies (1) and (2) induce a chain homotopy

  $\varrho_{2,0} : (A \otimes B)^{\otimes 4} \longrightarrow R$ such that

  $$\left(\varrho_{2,0} \circ d\right)(a|b, c|d, e|f, g|h) =$$

  $$\langle \varphi_3 (a|b, c|d, e|f), g|h \rangle - \langle \varphi_3 (c|d, e|f, g|h), a|b \rangle$$

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- $\exists$ additional bimodule structure s.t. $\left(A \otimes B, \{\varphi_n\}, \left\{\varrho_{k,l}\right\}\right)$ is an *A_∞-algebra with homotopy inner products* (HIPs)

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  2. a compatible family of higher inner products

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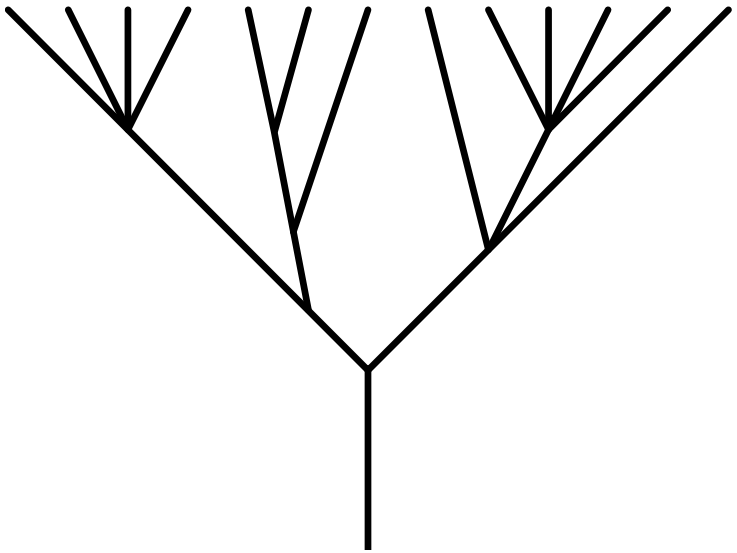
- Structure relations encoded by a *3-colored operad* $C_*\mathcal{A}$
- Identified with cellular chains of contractible *pairahedra*

The 3-colored operad CA

$C_*\mathcal{A}$ is generated by three types of planar diagrams

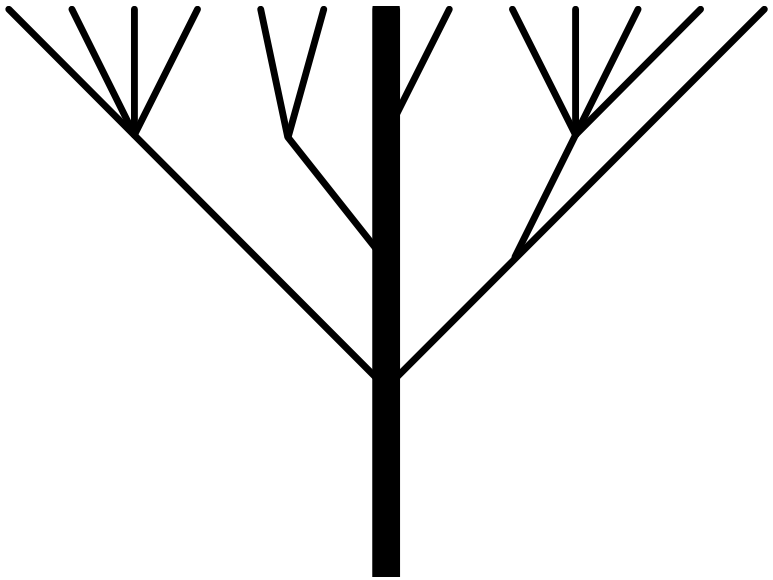
Colors of leaves and root: Empty, thin, thick

1. **Planar trees:** Control A_∞-algebra structure
   - *Thin leaves and root*



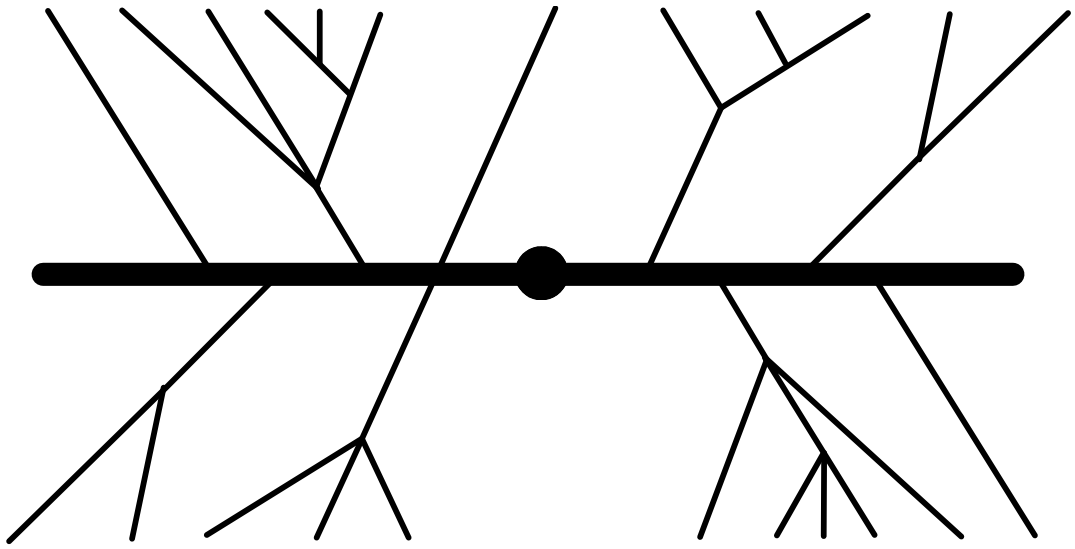
The 3-colored operad *CA*

2. **Module trees:** Control homotopy bimodule structure
   - *Thick vertical root and leaf*
   - *j thin leaves in left half-plane*
   - *k thin leaves in right half-plane*



The 3-colored operad *CA*

3. **Inner product diagrams:** Control HIP structure
   - *Empty root and two thick horizontal leaves*
   - *j thin leaves in upper half-plane*
   - *k thin leaves in lower half-plane*



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- Two inner product diagrams cannot be composed

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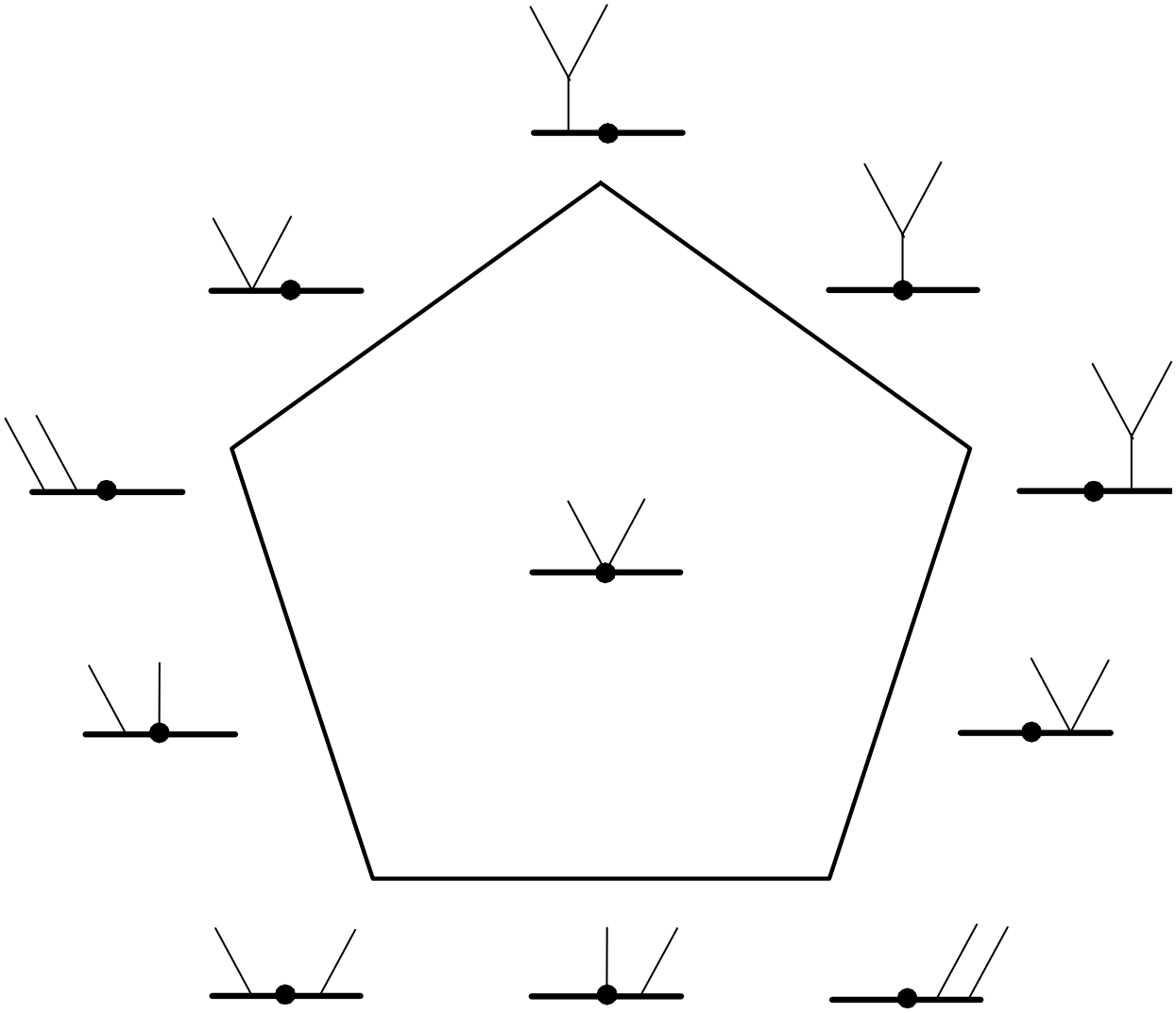
- **Example**: $C_* \mathrm{A}_1^{11\cdots 1}$ is generated by planar trees

Example

$C_* A_0^{2112}$ generated by IP diagrams $\leftrightarrow$ faces of pairahedron $I_{2,0}$:



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- $Q_*\mathcal{A}$ is a 3-colored operad

- $Q_*\mathcal{A}$ is generated by all *metric diagrams* (D, g), where

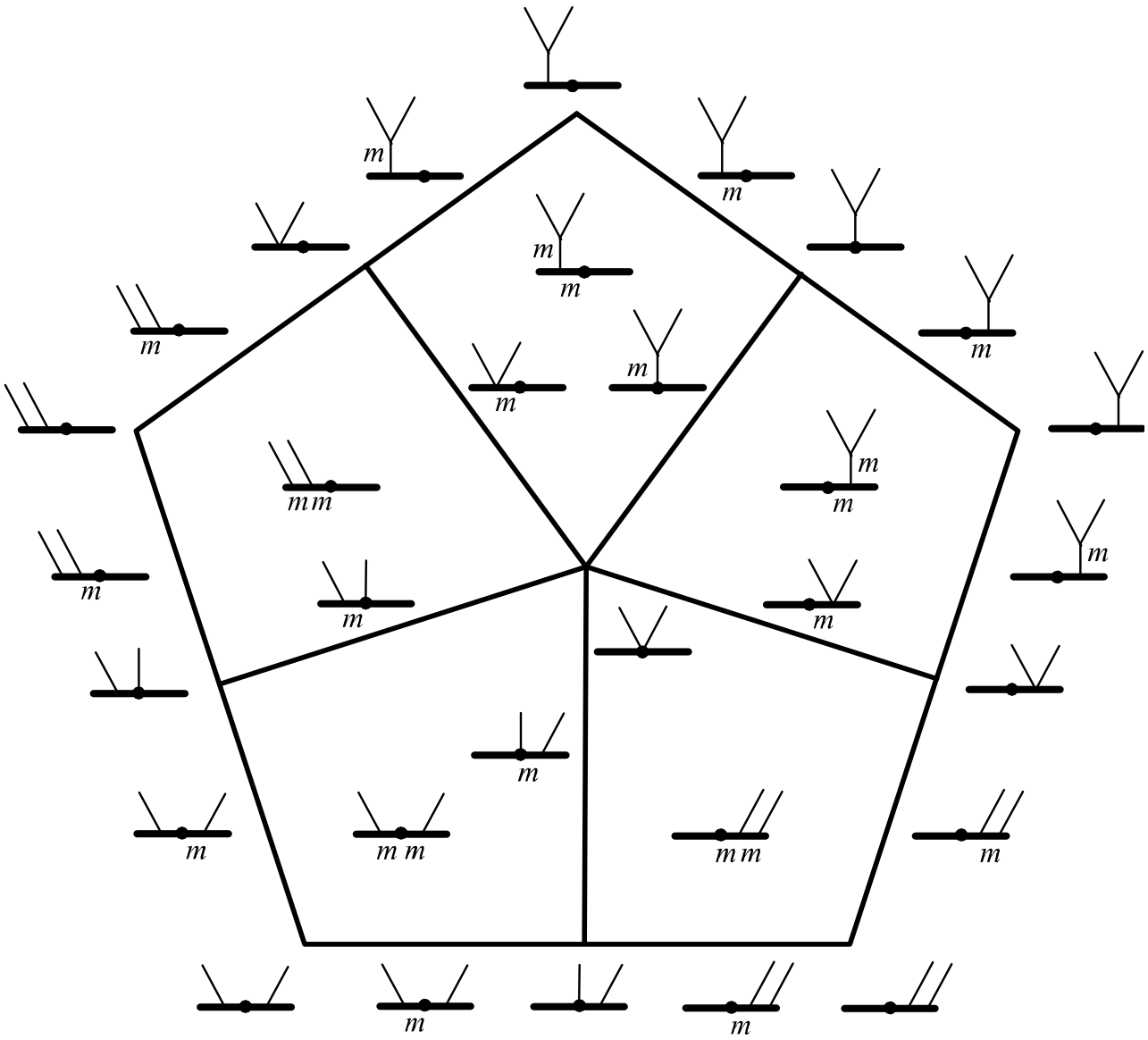
  - D is a generator of $C_*\mathcal{A}$

  - $g : \mathcal{E}(D) \to \{m, n\}$ labels the (internal) edges of D

  either "m" (metric) or "n" (non-metric)

Example

Cubical subdivision of $I_{2,0}$ (only metric labels are displayed):



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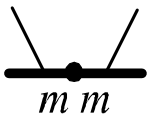
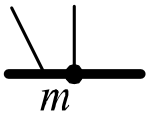
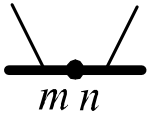
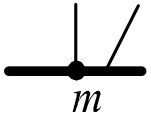
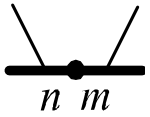
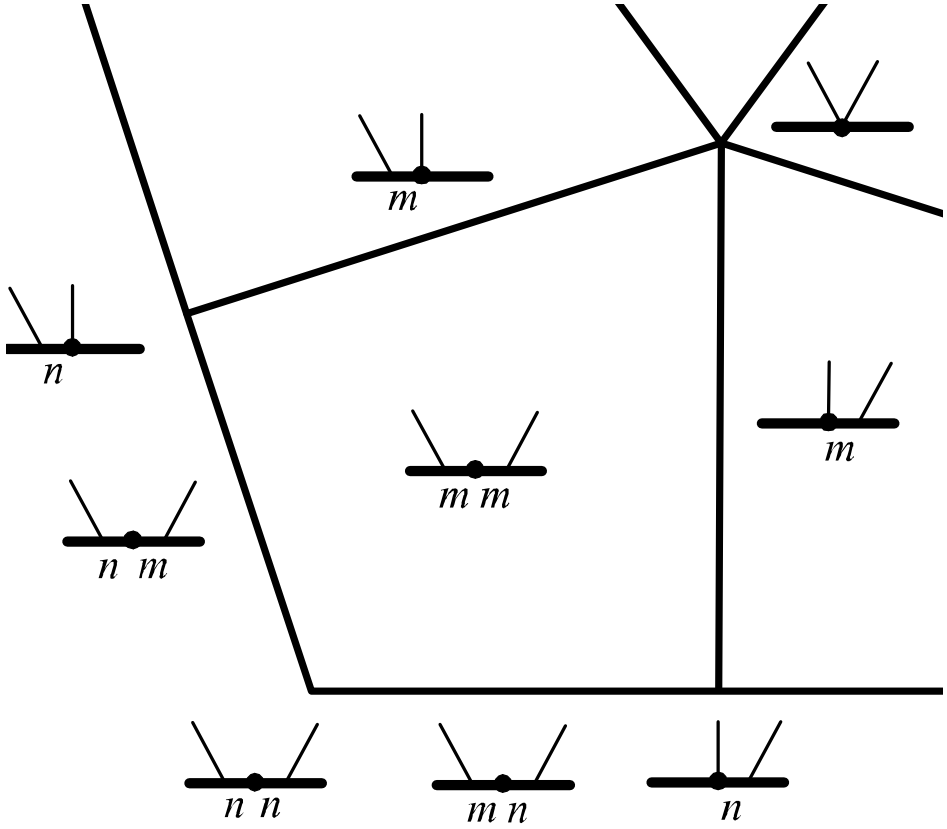
- $\partial_Q($  $=$  $+$  $+$  $+$ 

Example - Boundary of a Metric Square



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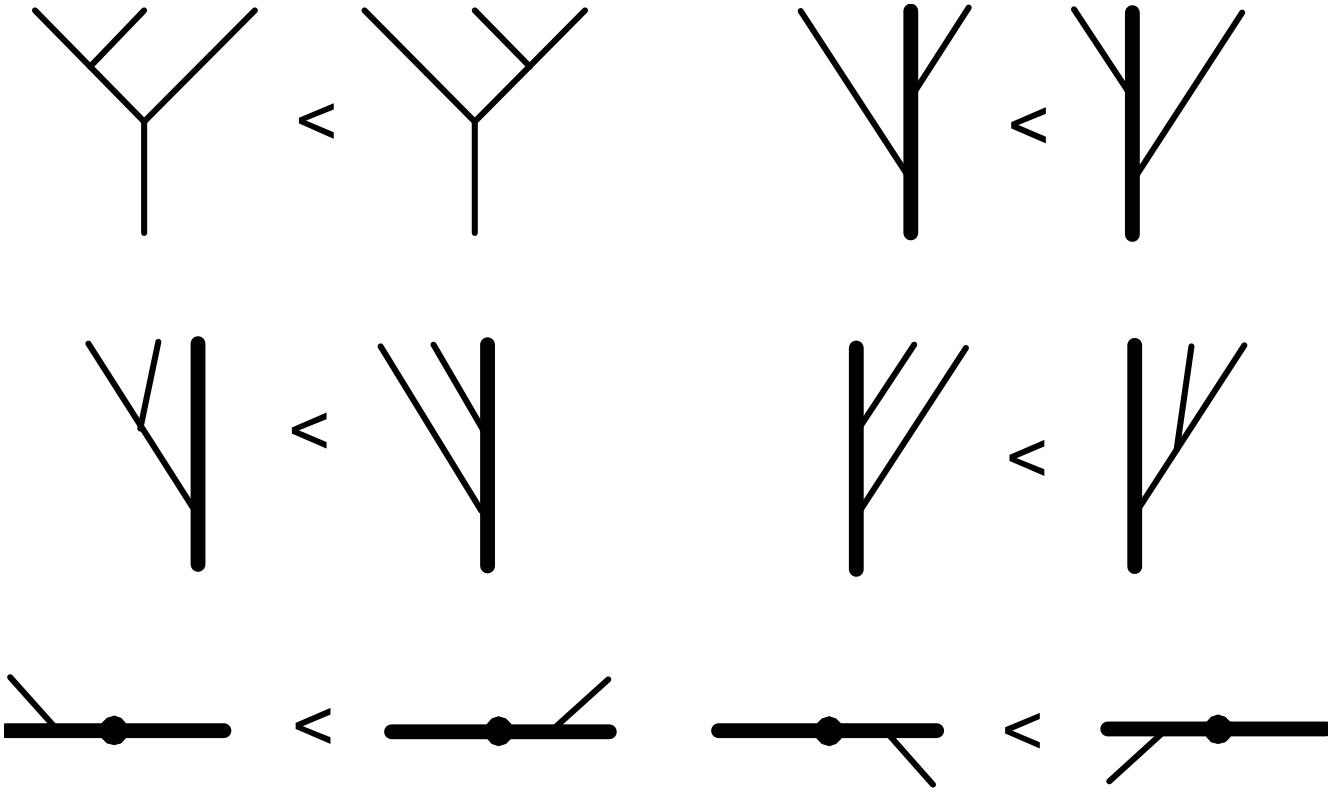
The Poset of Binary Diagrams in *CA*

- Extend Tamari ordering on binary trees to binary diagrams



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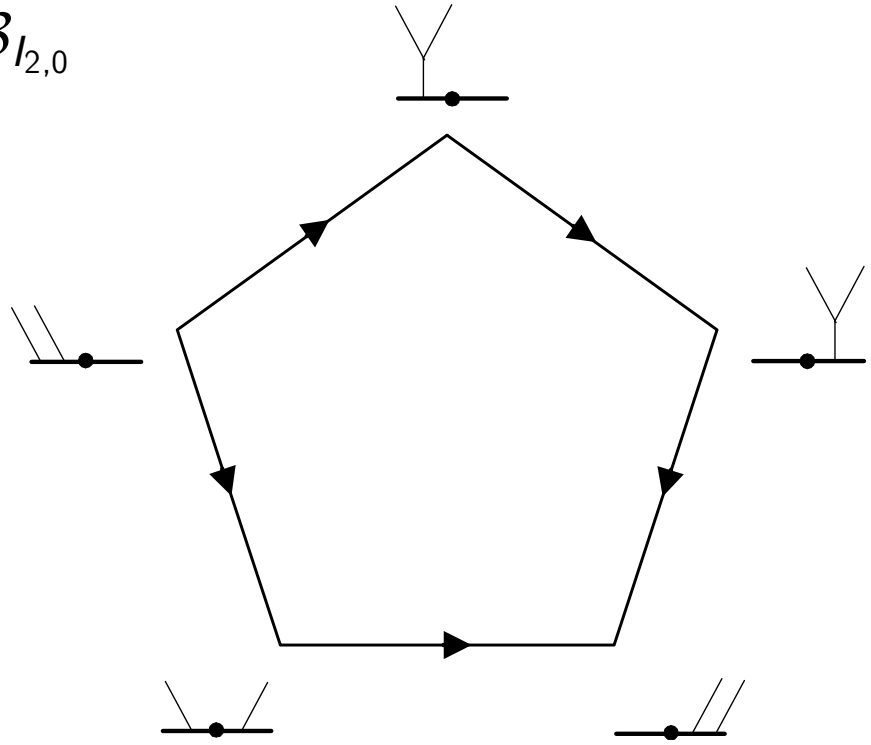

The 2-Sided Homotopy Inverse $p : QA \longrightarrow CA$

- **Definition.** On a *fully metric* $(D, m) \in Q_k \mathcal{A}^{\mathbf{x}}_y$ define

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$$p\,(B, m) = \begin{cases} c, & \text{if } B = c_{\max} \text{ for some corolla } c \\ 0, & \text{otherwise} \end{cases}$$

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- **Theorem**

1. *p and q* are *chain maps*
2. $pq = \mathrm{Id}$ *and* $qp \simeq \mathrm{Id}$

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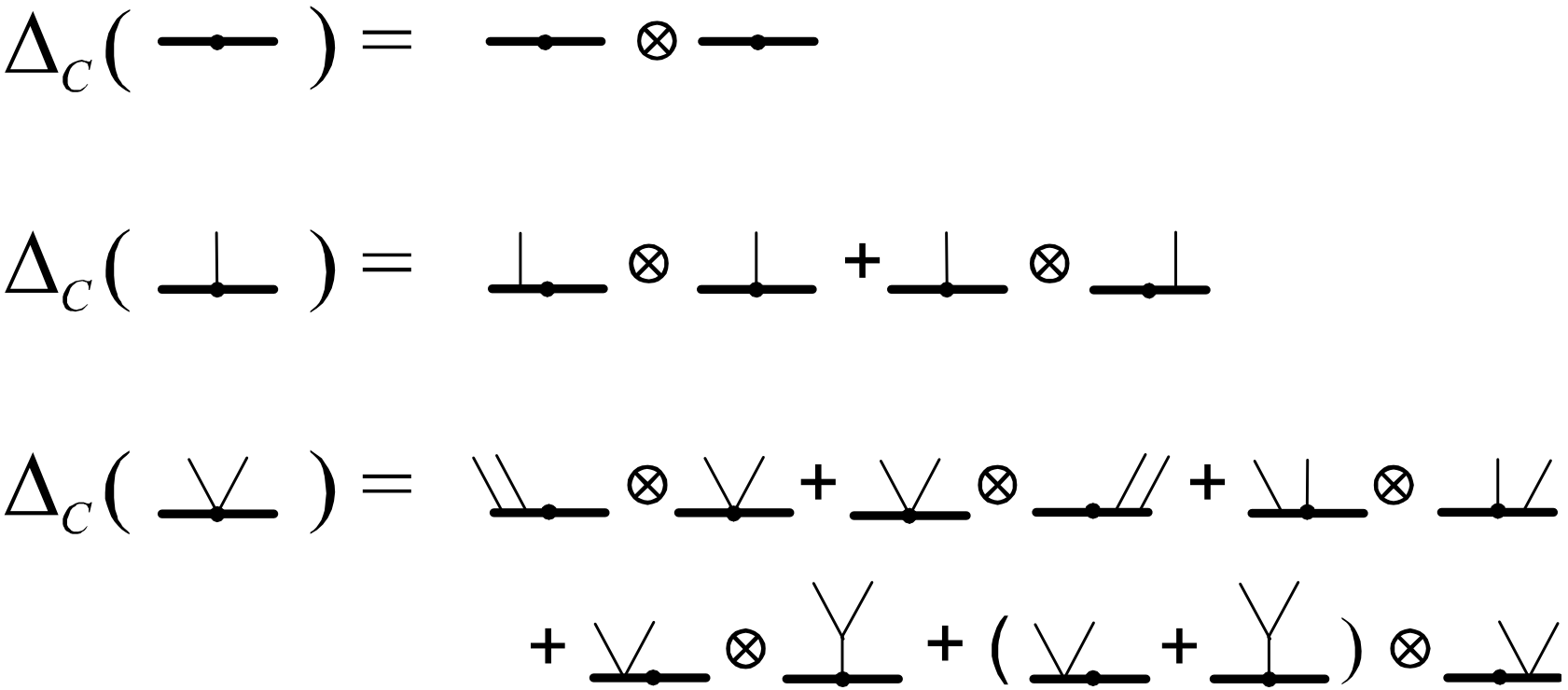
$$\Delta_C(c) = \sum_{\substack{S \otimes T \in C_i\mathcal{A}_y^{\mathbf{x}} \otimes C_{k-i}\mathcal{A}_y^{\mathbf{x}} \\ S_{\max} \leq T_{\min}}} S \otimes T$$

Examples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- Paper to appear in TAMS

Thank you!