The Saneblidze-Umble Diagonal on Associahedra

Presented by

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MU/F&M Joint Mathematics Colloquium

February 28, 2008

The Permutahedron P_n

- $P_n = \text{convex hull}\,\{(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n \mid \sigma \in S_n\}$
- Vertices $v_1, \ldots, v_{n!}$ in the hyperplane $x_1 + \cdots + x_n = \binom{n}{2}$
- P_n is an $(n-1)$-dimensional convex polyhedron

Combinatorics of P_n

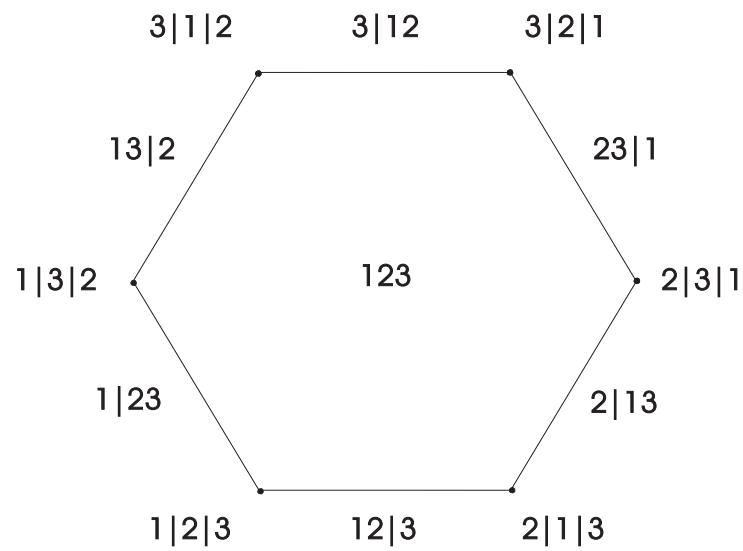
- Let $\underline{n} = \{1, 2, \ldots n\}$
- $\{$Faces in codim $p\} \leftrightarrow \{$Partitions $U_1 | \cdots | U_{p+1}$ of $\underline{n}\}$

  P_1 is a point $* \leftrightarrow \{1\}$

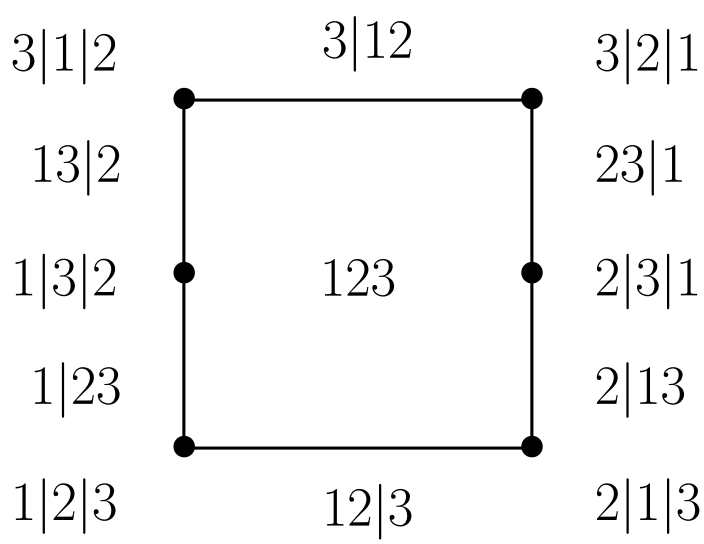
  P_2 is a closed interval I

  edge $\leftrightarrow \{12\}$; vertices $\leftrightarrow \{1|2, 2|1\}$

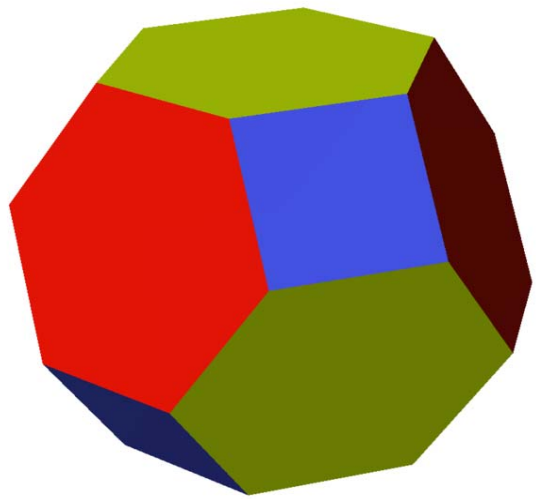
P_3 is a plane hexagonal region:



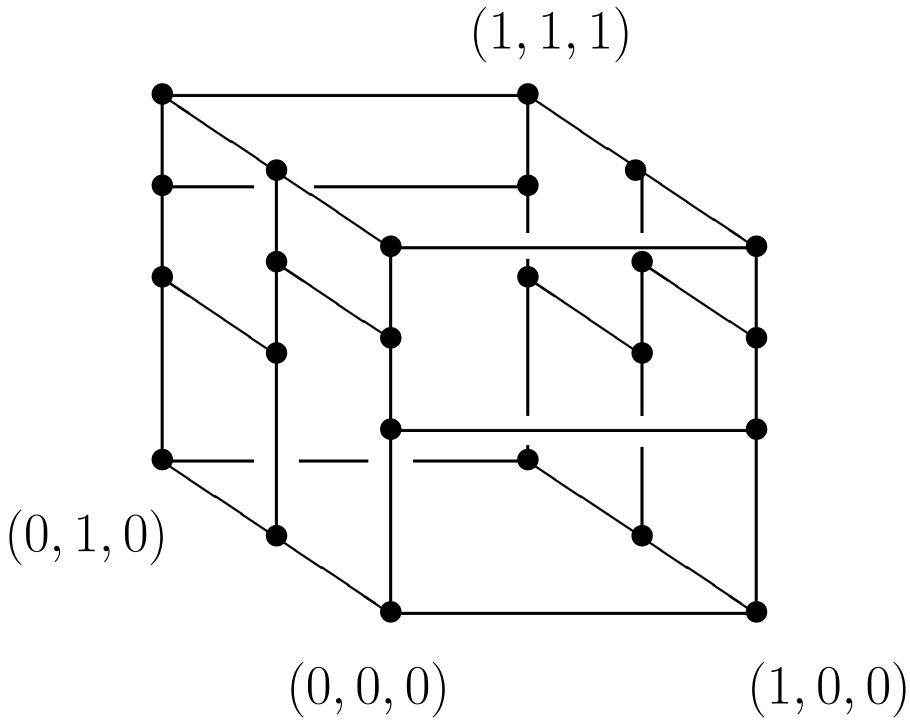
and can be obtained by subdividing $P_2 \times I$:



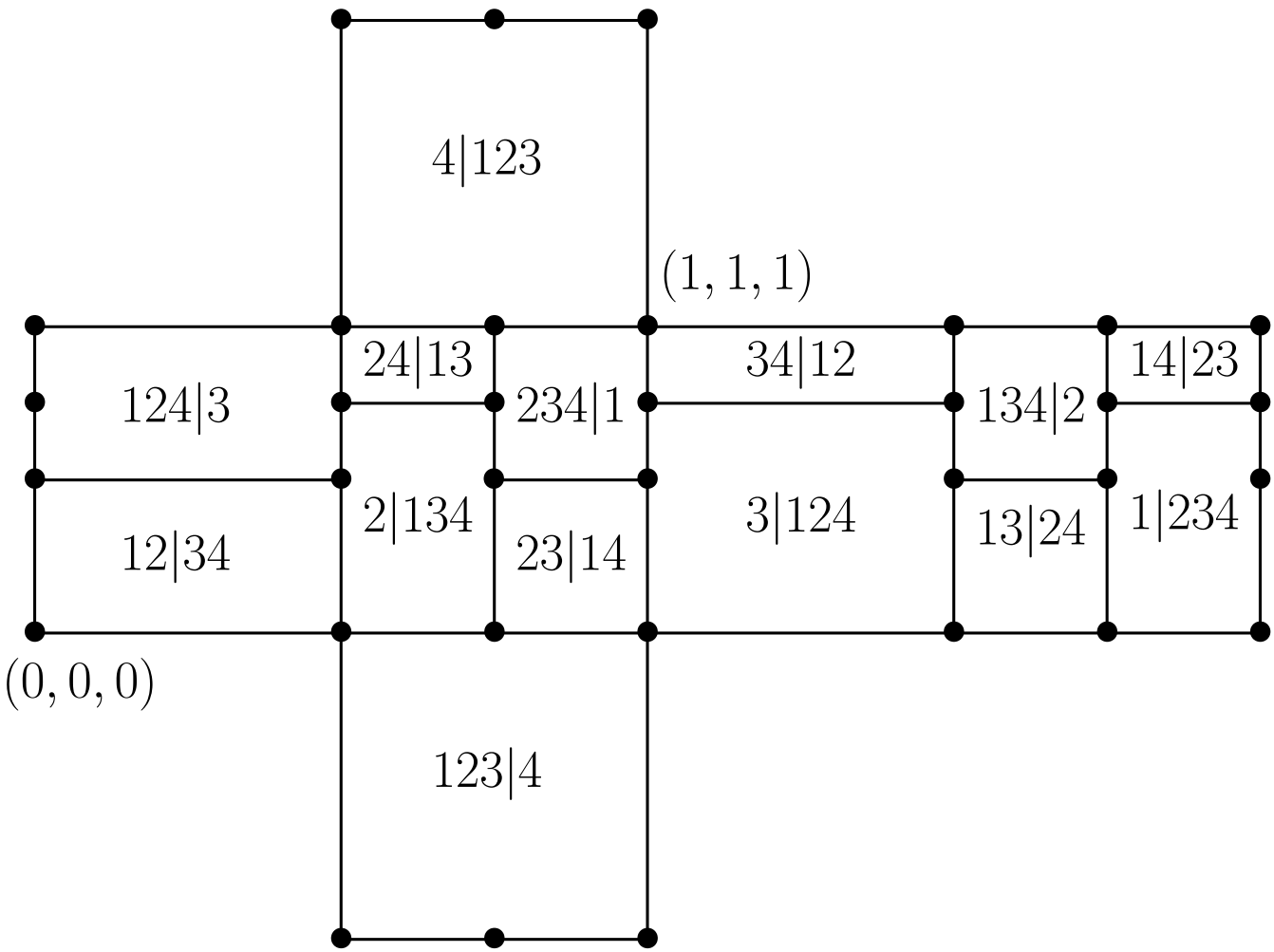
P_4 is a truncated octahedron:



and can be obtained by subdividing $P_3 \times I$:



The 2-faces of P_4:



Inductively, P_{n+1} can be obtained by subdividing $P_n \times I$

Cellular Chains

- Fix a ground field F; let X be a cellular complex
- *Cellular k-chains* of X are elements of the F-vector space $C_k(X)$ generated by the k-cells of X
- The *graded vector space of cellular chains of* X is

$$C_*(X) = \bigoplus_{k \geq 0} C_k(X)$$

- *Cellular boundary* ∂ is defined on the k-cells of X and extends to a linear map

$$\partial : C_*(X) \to C_{*-1}(X)$$

that satisfies $\partial \circ \partial = 0$.

- A *differential* on a graded vector space V_* is a linear map $d : V_* \to V_{*-1}$ such that $d \circ d = 0$
- ∂ is a differential on $C_*(X)$

- (V_*, d) is a *differential graded* (DG) *vector space*

- $(C_*(X), \partial)$ is DG vector space

- Let (V_*, d_V) and (W_*, d_W) be DG vector spaces.

  A linear map $f : V_* \to W_*$ is a *chain map* if

$$d_W \circ f = f \circ d_V$$

- Every map $f : X \to Y$ of cellular complexes is homotopic to a cellular map $g : X \to Y$.

- A cellular map $g : X \to Y$ induces a chain map $g : C_*(X) \to C_*(Y)$ by restricting to the cells of X and extending linearly

Diagonal Approximations

- The *geometric diagonal* $\Delta : X \to X \times X$ is

  defined by $\Delta(x) = x \times x$

- A *diagonal approximation* of Δ is a cellular map

  $\Delta_X : X \to X \times X$ homotopic to Δ

- Faces of the n-simplex $\Delta^n = [012 \cdots n]$ are

  indexed by all (increasing) subsequences

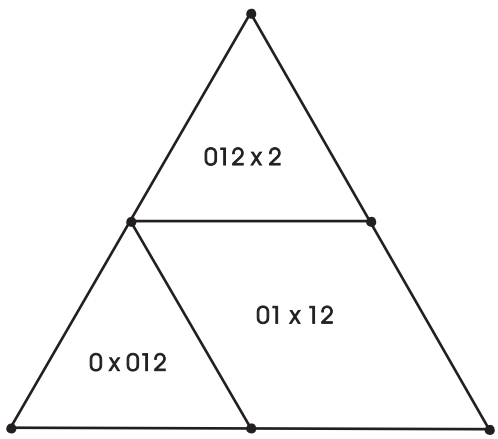
$$[a_0 \cdots a_k] \subseteq [012 \cdots n]$$

- $\partial([a_1 \cdots a_k]) = \sum (-1)^i [a_0 \cdots \widehat{a_i} \cdots a_k]$

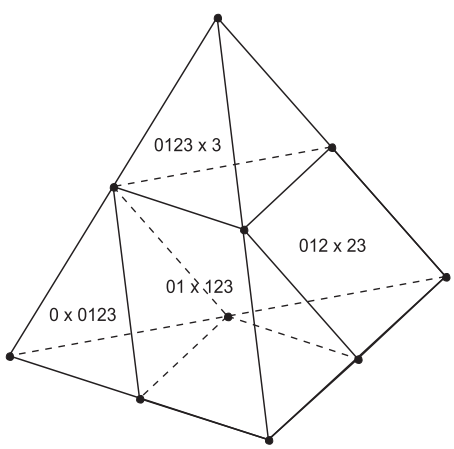
- Alexander-Whitney diagonal approximation:

$$\Delta_{\Delta^n}([012 \cdots n]) = \sum_{i=0}^{n} [01 \cdots i] \times [i \cdots n]$$

- A diagonal approx Δ_X determines a tessellation of X that transforms Δ_X into an inclusion



Tessellation of Δ^2 determined by A-W



Tessellation of Δ^3 determined by A-W

- Free linear extension of ∂ to $C_*(X) \otimes C_*(X)$ is a differential $\partial \otimes 1 + 1 \otimes \partial$

- A *diagonal* on $C_*(X)$ is a chain map

$$\Delta_X : C_*(X) \to C_*(X \times X) \approx C_*(X) \otimes C_*(X)$$

- A-W is strictly coassociative and homotopy cocommutative on cellular chains

The Saneblidze-Umble Diagonal Δ_P

- Step matrices

  A matrix E is a *step matrix* if for some n

  - Each element of $\underline{n}$ appears as an entry of E exactly once
  - The elements of $\underline{n}$ in each row and column of E form an increasing contiguous block
  - Each diagonal parallel to the main diagonal of E contains exactly one element of $\underline{n}$

  A typical step matrix for $n = 9$:

$$E = \begin{array}{|c|c|c|c|}\hline & & & 2 \\ \hline & & & 5 \\ \hline & & 4 & 6 \\ \hline 1 & 3 & 8 & \\ \hline 7 & & & \\ \hline 9 & & & \\ \hline \end{array}$$

- Right-shift operator on $G^{q\times p} = (g_{i,j})$

  Let $M_j \subset \{g_{*,j}\}$

  If $M_j \neq \varnothing$, then $\min M_j = g_{k,j}$ for some k

  If $\min M_j > \max\{g_{*,j+1}\}$ and

  $g_{t,j+1} = 0$ for $k \leq t \leq q$, obtain $R_{M_j}G$

  by interchanging $g_{i,j} \in M_j$ and $g_{i,j+1}$

  Otherwise, define $R_{M_j}G = G$

- Down-shift operator on $G^{q\times p} = (g_{i,j})$

  Let $N_i \subset \{g_{i,*}\}$

  If $N_i = \varnothing$, then $\min N_i = g_{i,k}$ for some k

  If $\min N_i > \max\{g_{i+1,*}\}$ and

  $g_{i+1,t} = 0$ for $k \leq t \leq p$, obtain $D_{N_i}G$

  by interchanging $g_{i,j} \in N_i$ and $g_{i+1,j}$

  Otherwise, define $D_{N_i}G = G$.

- Derived matrices

$$F = D_{N_p} D_{N_{p-1}} \cdots D_{N_1} R_{M_q} R_{M_{q-1}} \cdots R_{M_1} E$$

is a *derived matrix* iff E is a step matrix

- Step matrices are derived matrices via $M_i = N_j = \varnothing$

A typical derived matrix:

$$E = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & 5 & \\ \hline 4 & & \\ \hline \end{array} \quad \rightarrow \quad D_\varnothing D_\varnothing R_5 R_\varnothing E = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & & 5 \\ \hline 4 & & \\ \hline \end{array}$$

- Complementary pairs

$A_1|A_2|\cdots|A_p \times B_q|B_{q-1}|\cdots|B_1$ is a *complementary pair* (CP) of partitions iff B_i and A_j are the rows and columns of a $q \times p$ derived matrix

  - The CP $14|25|3 \times 4|15|23 \leftrightarrow E$ above

- $\{$Derived matrices$\} \leftrightarrow \{$CPs$\}$
- $\{$CPs$\} \leftrightarrow \{$Faces of $P_n \times P_n\}$

- The S-U diagonal Δ_P (2004)

  Let e^{n-1} be the top dim'l face of P_n

  Define $\Delta_P(e^0) = e^0 \otimes e^0$.

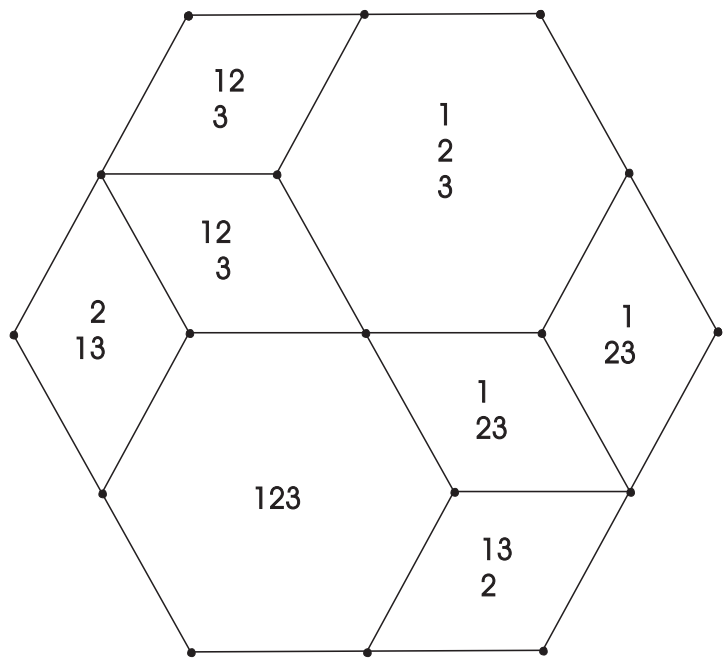
  Having defined Δ_P on $C_*(P_k)$, $k \leq n-1$,

  define Δ_P on $C_{n-1}(P_n)$ by

$$\Delta_P(e^{n-1}) = \sum_{\substack{(p,q)\text{-CPs } u\times v \\ p+q=n+1}} \pm\, u \otimes v$$

  and extend multiplicatively to $C_*(P_n)$

$\Delta_P(P_3)$:



The Associahedron K_n (Stasheff 1963)

- $K_n = P_{n-1} / \sim$
- $\{$Codim p faces of $P_{n-1}\} \leftrightarrow$

  $\{$PLTs with n leaves and $p+1$ levels$\}$

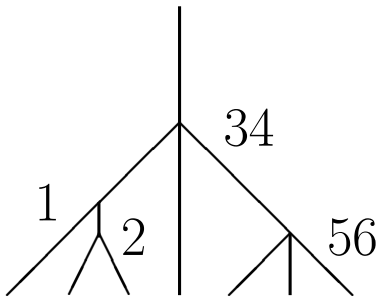
  (PLT = Planar rooted Leveled Tree)
- $\{$PLTs with n leaves and $p+1$ levels$\} \leftrightarrow$

  $\{$Partitions of $\underline{n-1}$ of length $p+1\}$

  Given a PLT T, number the leaves from left to right and assign the label i to the node at which the branch of leaf i meets the branch of leaf $i+1$. Then

  $$T \leftrightarrow U_1 | \cdots | U_{p+1},$$

  where $U_j = \{$labels of nodes in level $j\}$



$\leftrightarrow \quad 256|1|34$

- $\{$Faces of $K_n\} \leftrightarrow \{$PRTs with n leaves$\}$

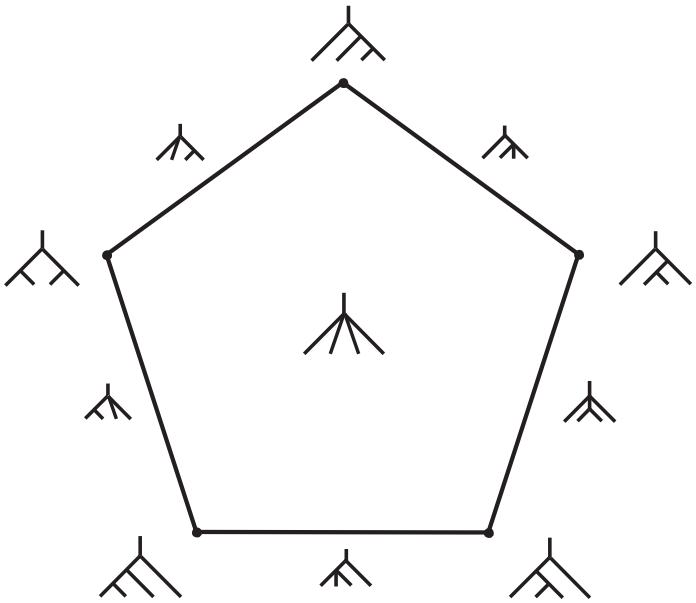
  (PRT = Planar Rooted Tree)

  K_2 is a point $* \leftrightarrow \{$ $\}$

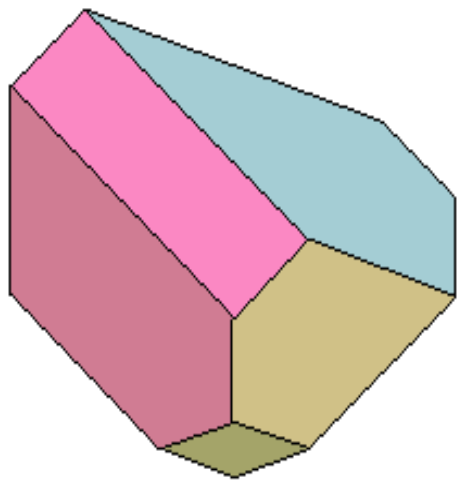
  K_3 is a closed interval I

  edge $\leftrightarrow \{$  $\}$; vertices $\leftrightarrow \{$  ,  $\}$

  K_4 is a plane pentagonal region:



K_5:



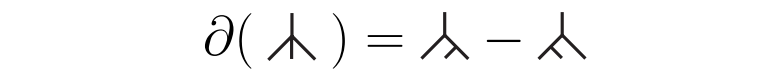
- $\{$PRTs with n leaves$\}$ $\leftrightarrow$

  $\{$Parenthesizations of n variables$\}$

- $\partial : C_*(K_n) \to C_{*-1}(K_n)$ is defined by

$$\partial(T) = \sum \pm T_i,$$

  where T_i has one more node than T and the T_i's range over all possible trees obtained from T by grafting in a branch at some node of valance > 3



- Forgetting levels defines Tonks' projection (1997)

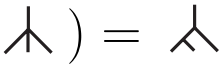
$$\theta : P_{n-1} \to K_n$$

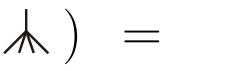
- Faces of P_{n-1} indexed by PLTs with multiple nodes in the same level degenerate under θ; corresponding generators lie in the kernel of the induced map $\theta : C_*(P_{n-1}) \to C_*(K_n)$

- The S-U diagonal Δ_K is defined by

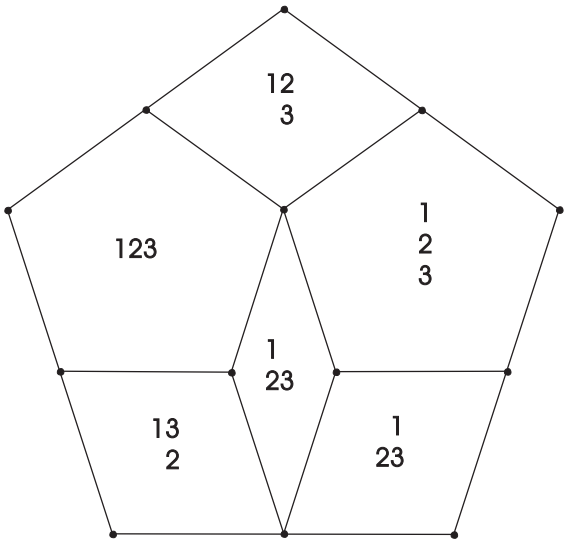
$$\Delta_K \theta = (\theta \otimes \theta)\Delta_P$$

- $\Delta_K($  $=$ $\otimes$

- $\Delta_K($  $=$ $\otimes$  $+$  $\otimes$ 

- $\Delta_K($  $=$  $\otimes$  $+$  $\otimes$  $+$  $\otimes$ 

$+$  $\otimes$  $+$  $\otimes$  $+$  $\otimes$ 



A_∞**-algebras** (Stasheff 1963)

- Let (A, d) be a DG vector space and consider

$$\left\{ Hom^p \left(A^{\otimes n}, A \right) \right\}_{p \in \mathbb{Z}}$$

- Free linear extension of d to $A^{\otimes n}$ is a differential

$$d_n \;=\; d \otimes 1^{\otimes n-1} + 1 \otimes d \otimes 1^{n-2} + \cdots + 1^{\otimes n-1} \otimes d$$

- d induces a differential

$$\delta : Hom^p \left(A^{\otimes n}, A \right) \to Hom^{p-1} \left(A^{\otimes n}, A \right) \quad \text{via}$$

$$\delta \left(f \right) \;=\; d \circ f - (-1)^p \, f \circ d_n$$

- $\left(Hom^* \left(A^{\otimes n}, A \right), \delta \right)$ is a DG vector space

- Identify the top dimensional cell e^{n-2} of K_n with a multilinear operation $\mu_n \in Hom^{n-2}\left(A^{\otimes n}, A\right)$

- An A_∞-*algebra* is a DG vector space (A, d) together with a family of operations

$$\left\{\mu_n \in Hom^{n-2}\left(A^{\otimes n}, A\right)\right\}_{n\geq 2}$$

and a family of chain maps

$$\left\{\zeta : C_*\left(K_n\right) \to Hom^{n-2}\left(A^{\otimes n}, A\right)\right\}_{n\geq 2}$$

such that $\zeta\left(e^{n-2}\right) = \mu_n$ and $\zeta\left(T\right)$ on a proper face $T \subset K_n$ is the composition of μ_k's specified by T

- Quadratic Structure Relations

Multiplication is homotopy associative and μ_3 is an associating homotopy:

$$\delta\mu_3 = \delta\zeta(\ \lambda\ ) = \zeta\partial(\ \lambda\ )$$

$$= \zeta(\ \lambda - \lambda\ )$$

$$= \mu_2\,(1 \otimes \mu_2) - \mu_2\,(\mu_2 \otimes 1)$$

Stasheff's Pentagon Condition:

$$\delta\mu_4 = \delta\zeta(\ \lambda\ ) = \zeta\partial(\ \lambda\ )$$

$$= \zeta(\ \lambda + \lambda + \lambda - \lambda - \lambda\ )$$

$$= \mu_2\,(\mu_3 \otimes 1) + \mu_3\,(1 \otimes \mu_2 \otimes 1)$$

$$+\mu_2\,(1 \otimes \mu_3) - \mu_3\,(\mu_2 \otimes 1 \otimes 1)$$

$$-\mu_3\,(1 \otimes 1 \otimes \mu_2)$$

Tensor Products of A_∞-algebras (S-U 2000)

- Let (A, ζ_A) and (B, ζ_B) be A_∞-algebras.

  The *induced A_∞-algebra structure* $\zeta_{A\otimes B}$ is given by the composition

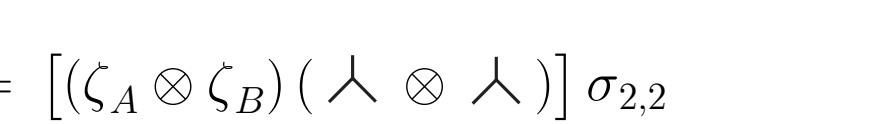
$$\begin{array}{ccc} C_*(K_n) & \xrightarrow{\zeta_{A\otimes B}} & Hom((A\otimes B)^{\otimes n}, A\otimes B) \\ {\scriptstyle \Delta_K}\downarrow & & \uparrow{\scriptstyle (\sigma_{2,n})^*} \\ C_*(K_n)\otimes C_*(K_n) & \xrightarrow[\zeta_A\otimes\zeta_B]{} & Hom(A^{\otimes n}\otimes B^{\otimes n}, A\otimes B) \end{array}$$

  where $\sigma_{2,n} : (A\otimes B)^{\otimes n} \to A^{\otimes n}\otimes B^{\otimes n}$ is the canonical permutation of tensor factors

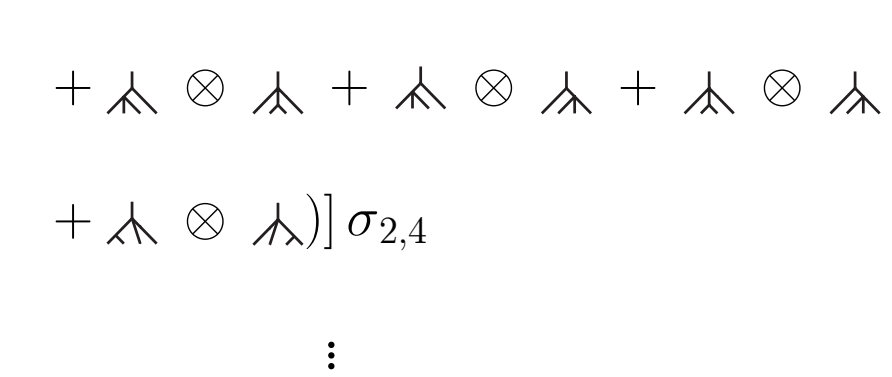
- The A_∞-algebra operations on $A \otimes B$ are

$$\Phi_n \;=\; \zeta_{A\otimes B}\left(e^{n-2}\right) = \left[\left(\zeta_A \otimes \zeta_B\right)\Delta_K\left(e^{n-2}\right)\right]\sigma_{2,n}$$

and in particular

$\Phi_2 =$ 

$\Phi_3 =$ 

$\Phi_4 =$ 

$\vdots \qquad\qquad \vdots$

Tensor Products of Simple A_∞-algebras

- A *simple A_∞-algebra* has exactly one non-trivial operation μ_n for $n \geq 3$
- Let $A = \Lambda(x, y)$ with $|x| = 1$, $|y| = 4$ and $d = 0$

  Define

$$\mu_3\left(x^i y^p | x^j y^q | x^k y^r\right) = \begin{cases} y^{p+q+r+1}, & ijk = 1 \text{ and } pqr \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

  All quadratic relations hold:

$$\mu_2(\mu_3 \otimes 1) + \mu_3(1 \otimes \mu_2 \otimes 1) + \mu_2(1 \otimes \mu_3)$$
$$= \mu_3(\mu_2 \otimes 1 \otimes 1) + \mu_3(1 \otimes 1 \otimes \mu_2)$$

$$\mu_3(\mu_3 \otimes 1 \otimes 1 + 1 \otimes \mu_3 \otimes 1 + 1 \otimes 1 \otimes \mu_3) = 0$$

  Therefore (A, μ, μ_3) is a simple A_∞-algebra

- Over $\mathbb{Z}_3$ there are two non-trivial higher order operations on $A \otimes A$, namely,

$$\Phi_3 = \left[\mu_2 \left(\mu_2 \otimes 1\right) \otimes \mu_3 + \mu_3 \otimes \mu_2 \left(1 \otimes \mu_2\right)\right] \sigma_{2,3}$$

$$\begin{aligned}\Phi_4 = \big[&\mu_2 \left(\mu_3 \otimes 1\right) \otimes \mu_3 \left(1 \otimes \mu_2 \otimes 1\right) \\ &+\mu_2 \left(\mu_3 \otimes 1\right) \otimes \mu_2 \left(1 \otimes \mu_3\right) \\ &+\mu_3 \left(1 \otimes \mu_2 \otimes 1\right) \otimes \mu_2 \left(1 \otimes \mu_3\right) \\ &+\mu_3 \left(\mu_2 \otimes 1 \otimes 1\right) \otimes \mu_3 \left(1 \otimes 1 \otimes \mu_2\right)\big]\sigma_{2,4}.\end{aligned}$$

Application to Group Cohomology

- $H^*(C_n; \mathbb{Z}_p)$ is an A_∞-algebra (Madson 2002)
- The A_∞-algebra structure of

$$H^*(C_m \times C_n; \mathbb{Z}_p) \approx H^*(C_m; \mathbb{Z}_p) \otimes H^*(C_n; \mathbb{Z}_p)$$

was computed by Mikael Vejdemo Johansson in his 2008 thesis. Mikael will arrive in Millersville on March 25 and be in residence for approximately one month. This discussion will continue in his colloquium talk on March 27.