Detecting Linkage in an n-Component Brunnian Link

IMUS Mini-Course Session 1

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Millersville U and IMUS

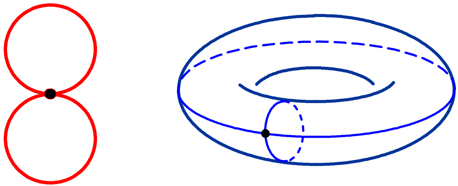
24 April 2018

Goal of the Project:

To computationally detect the linkage in an n-component Brunnian link

Review of Cellular Complexes

Let X be a connected network, surface, or solid embedded in S^3



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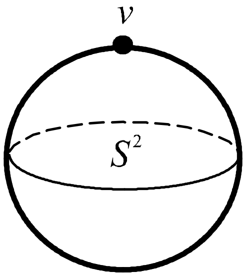
  - Non-empty intersection of cells is a cell
  - Union of all cells is X

Example: 2-dim'l Sphere

$S^2 = D^2/\partial D^2$ (Grandma's draw string bag)

- Vertex: $\{v\}$
- Edges: $\varnothing$
- Face: $\{S^2\}$

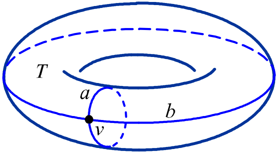


Example: Torus

$T = S^1 \times S^1$

Product cells: $\{v{,}a\} \times \{v{,}b\}$

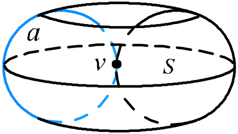
- Vertex: $\{v\!: = v \times v\}$
- Edges: $\{a := a \times v,\ b := v \times b\}$
- Face: $\{T := a \times b\}$



Example: Pinched Sphere

$P = T/b$

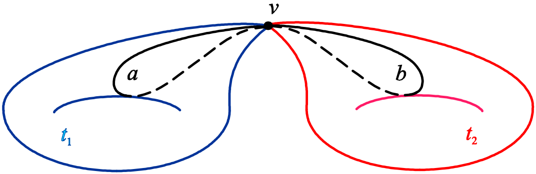
- Vertex: $\{v\}$
- Edge: $\{a\}$
- Face: $\{S\}$



Example: Link Complement of Two Unknots

Let UN be the complement of disjoint tubular neighborhoods U_1 and U_2 of **two unlinked unknots in** S^3

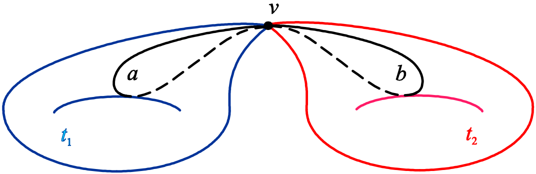
- $\partial(U_1 \cup U_2)$ is the wedge of two pinched spheres t_1 and t_2 with respective edges a and b and shared vertex v



$$\partial(U_1 \cup U_2) = \partial(UN)$$

Cellular Structure of UN

- $\partial\,(UN)$ is wedged with the equatorial 2-sphere $s \subset S^3$
- $p =$ upper hemispherical 3-ball
- $q =$ lower hemispherical 3-ball $\smallsetminus (U_1 \cup U_2)$
- p and q are attached along s
- $UN = p \cup q$

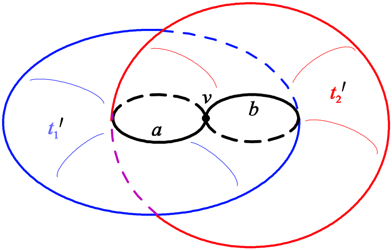


- Vertices: $\{v\}$
- Edges: $\{a, b\}$
- Faces: $\{s, t_1, t_2\}$
- Solids: $\{p, q\}$

Example: Link Complement of the Hopf Link

Let LN be the complement of disjoint tubular neighborhoods U_i of the **Hopf Link** in S^3

- $\partial (U_1 \cup U_2)$ is the union of two linked tori t_1' and t_2' sharing edges a and b and vertex v



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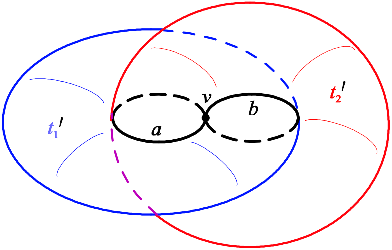

- Vertex: $\{v\}$
- Edges: $\{a, b\}$
- Faces: $\{s, t_1', t_2'\}$
- Solids: $\{p, q'\}$

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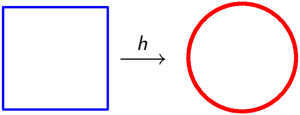

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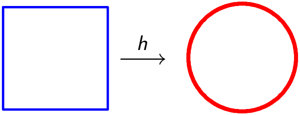

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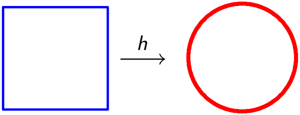

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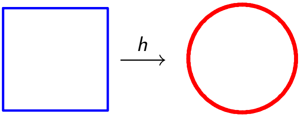

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- UN and LN are not homeomorphic because...

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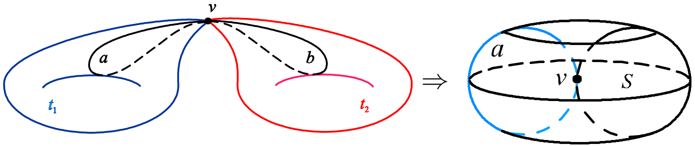
  - Shrinks $\partial(UN)$ to a pinched sphere



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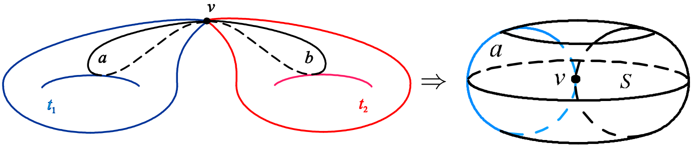

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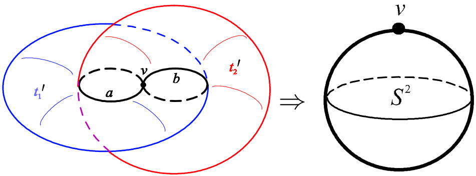

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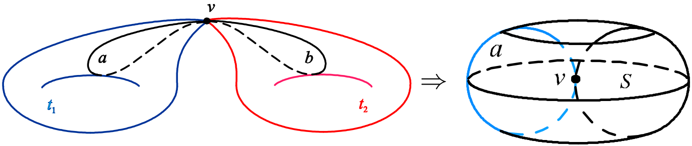

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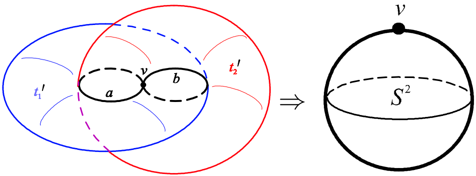

- How do we can detect this computationally?

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  - Show that h fails to respect diagonals

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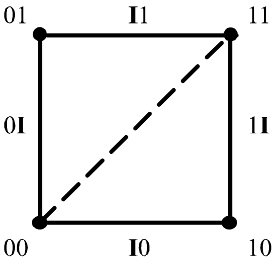
- **Example:** $\operatorname{Im}\Delta_{\mathtt{I}}$ is not a subcomplex of $\mathtt{I} \times \mathtt{I}$:



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- **Cellular Approximation Theorem**

There is a diagonal approximation $\Delta : X \to X \times X$

Properties of Diagonal Approximations

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- Cartesian products: $\Delta\left(X \times Y\right) = \Delta\left(X\right) \times \Delta\left(Y\right)$
- Wedge products: $\Delta\left(X \vee Y\right) = \Delta\left(X\right) \vee \Delta\left(Y\right)$

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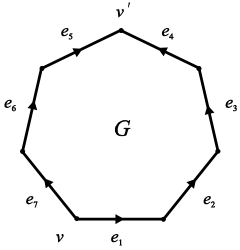

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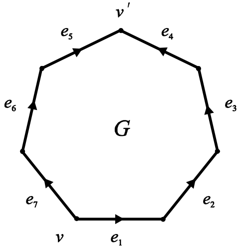
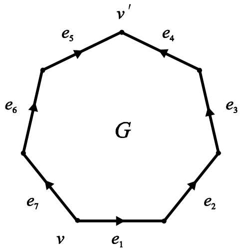


- **Theorem (Kravatz 2008):** *There is a diagonal approximation*

$$\Delta G = v \times G + G \times v' + \sum_{1 \leq i < j \leq k} e_i \times e_j + \sum_{n \geq j > i \geq k+1} e_j \times e_i$$

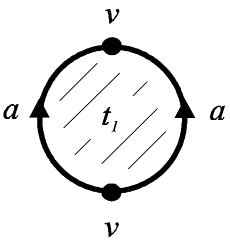
Example: The Heptagon G



$$\Delta G = v \times G + G \times v'$$
$$+e_1 \times (e_2 + e_3 + e_4) + e_2 \times (e_3 + e_4) + e_3 \times e_4$$
$$+e_7 \times (e_6 + e_5) + e_6 \times e_5$$

Example: The Pinched Sphere

Think of the **pinched sphere** $t_1 \subset \partial(UN)$ as a 2-gon with vertices identified first, then edges identified

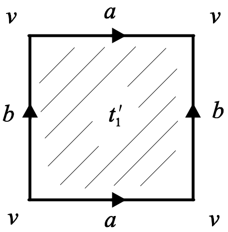


$$\Delta t_1 = v \times t_1 + t_1 \times v$$

- Δ descends to quotients when edge-paths are consistent with identifications

Example: The Torus

Think of the **torus** $t_1' \subset \partial(LN)$ as a square with horizontal edges a identified and vertical edges b identified



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Cellular Chains of a Space

- $C\left(X\right)$ denotes the $\mathbb{Z}_2$-vector space with basis $\{\text{cells of } X\}$
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- **Examples:**
  - $C\left(UN\right)$ has basis $\{v, a, b, s, t_1, t_2, p, q\}$
  - $C\left(LN\right)$ has basis $\{v, a, b, s, t_1', t_2', p, q'\}$
  - Note that $C\left(UN\right) \approx C\left(LN\right)$

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- **The boundary operator** $\partial : C(X) \to C(X)$ is
  - Induced by the geometric boundary
  - Zero on vertices
  - Linear on chains
  - A derivation of the Cartesian product

$$\partial(a \times b) = \partial a \times b + a \times \partial b$$

Examples

- $\partial : C(UN) \to C(UN)$ is defined

$$\partial v = \partial a = \partial b = \partial s = \partial t_1 = \partial t_2 = 0$$
$$\partial p = s$$
$$\partial q = s + t_1 + t_2$$

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$$\begin{aligned}
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- How do diagonal approximations on UN and LN descend to homology?

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- *Assume* $h : UN \to LN$ *is a homeomorphism; show that* $\Delta_2 h_* \neq (h \times h)_* \Delta_2$

Homology of Cartesian Products

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- Since $\mathbb{Z}_2$ is a field, torsion vanishes and

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Induced Diagonal on H(X)

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- **Examples**

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- **The non-primitive coproduct has detected the Hopf Link!**
- **Goal: Apply this strategy to n-component Brunnian Links**

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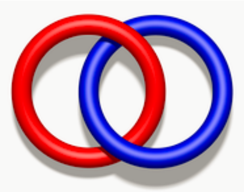

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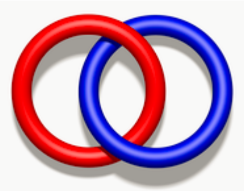
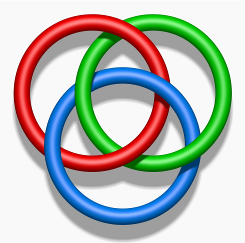
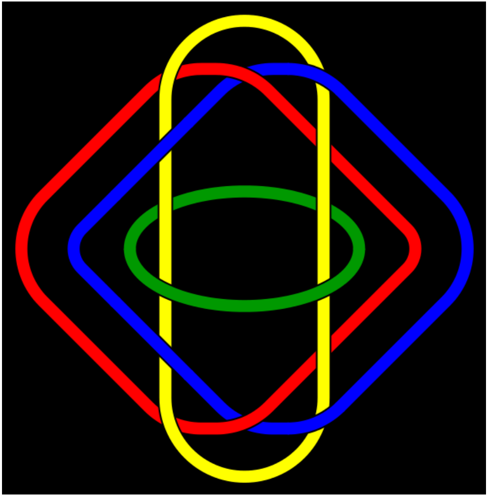


- The most familiar example is the **Borromean rings**



Brunnian Links

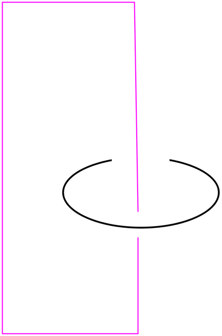
- A 4-component **Brunnian link**



Brunnian Links

An animated 6-component Brunnian link

The Hopf link: A Brunnian link with two components



Constructing a Brunnian link with 3 components

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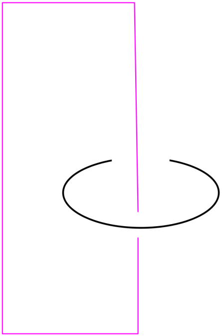

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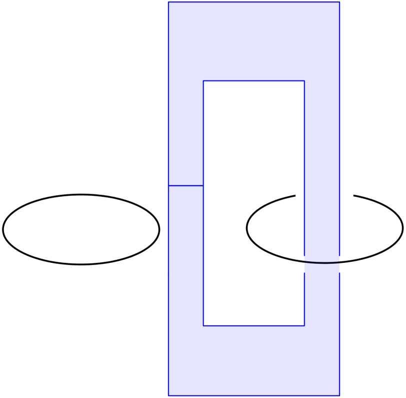

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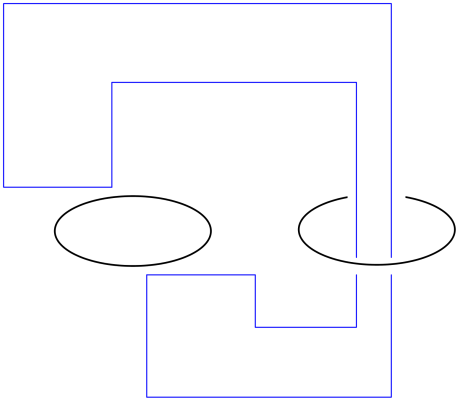

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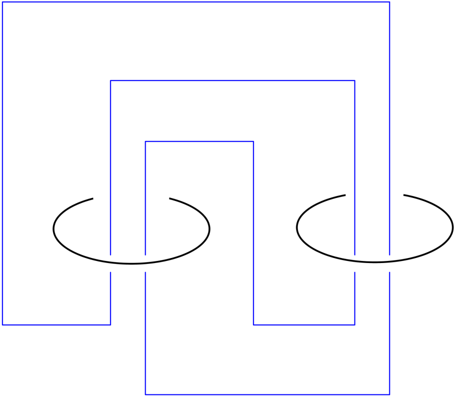

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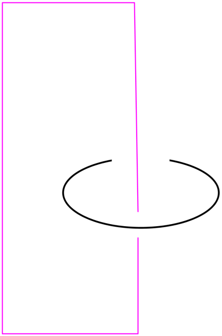

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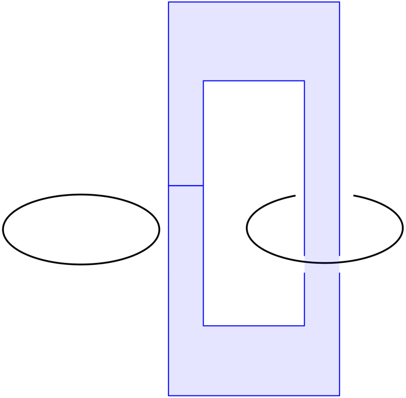

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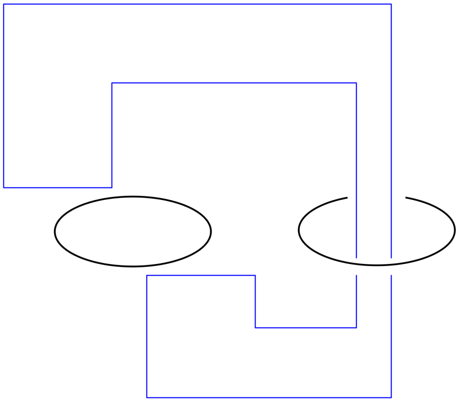

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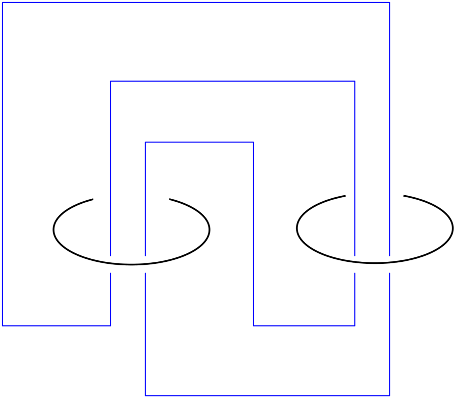

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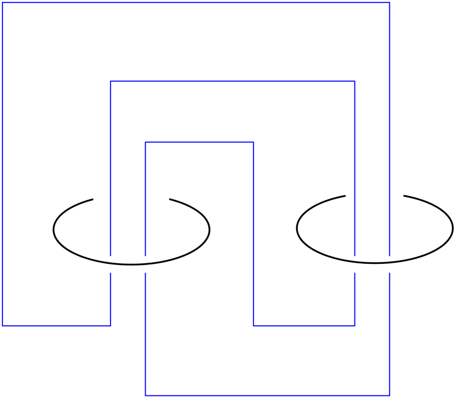

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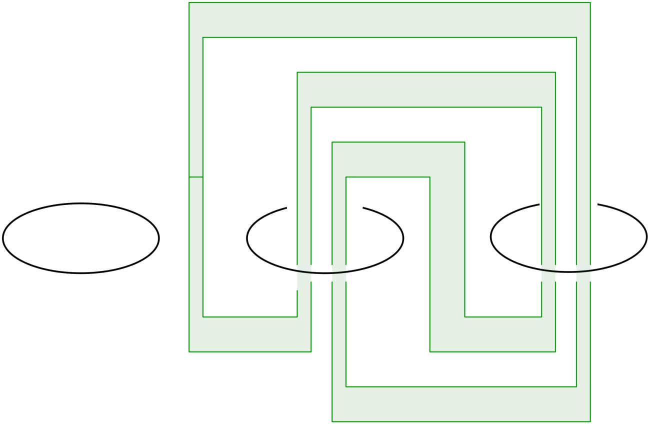

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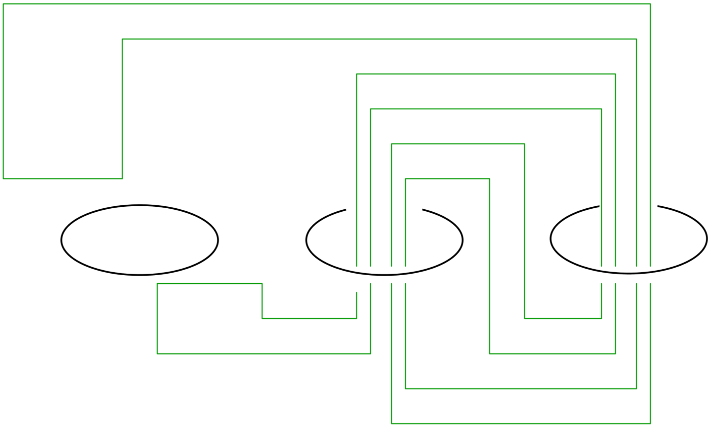

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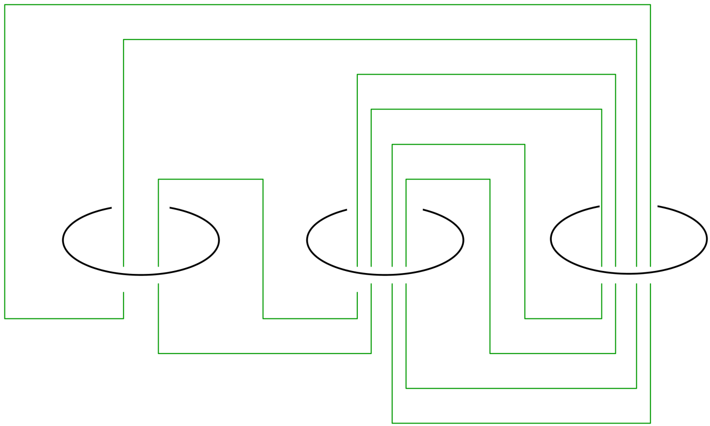

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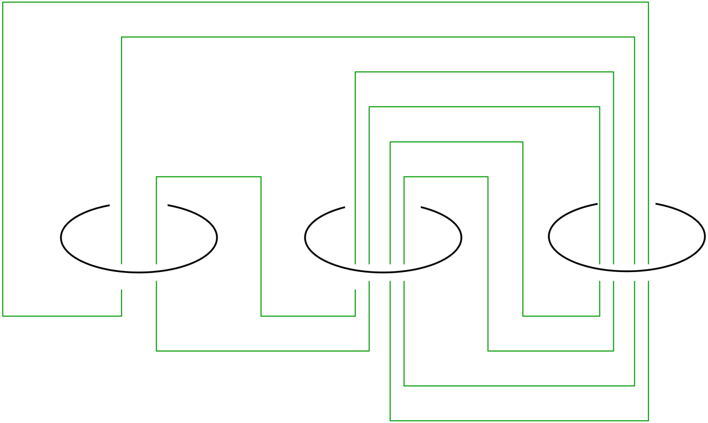

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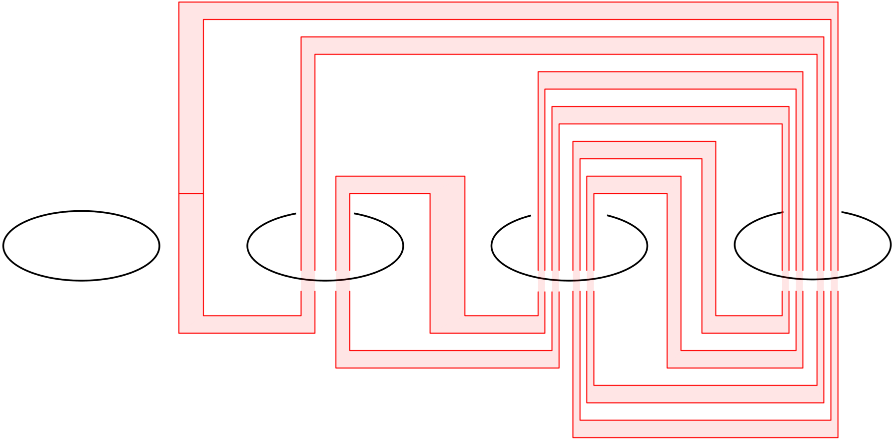

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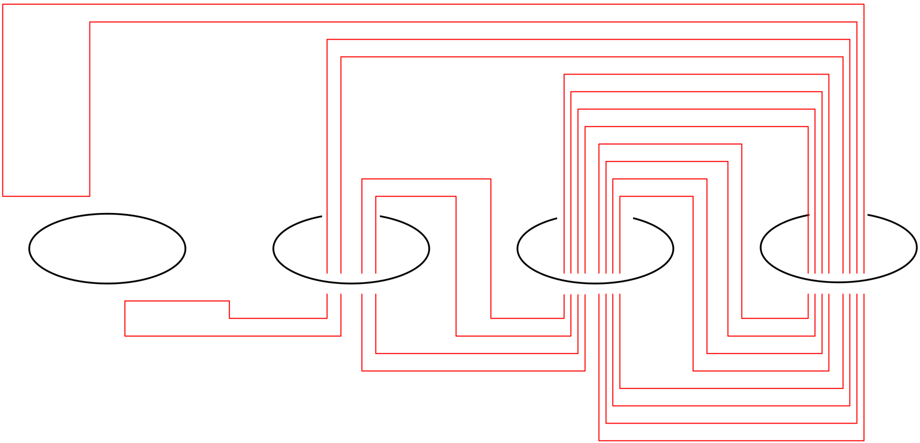

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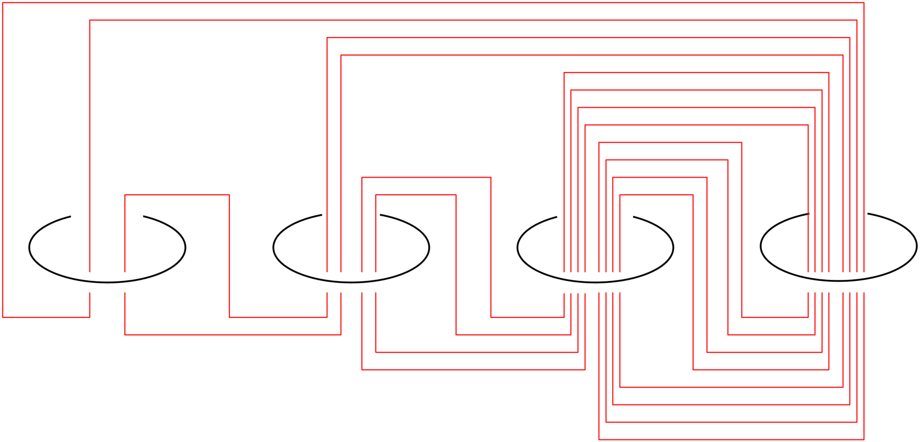

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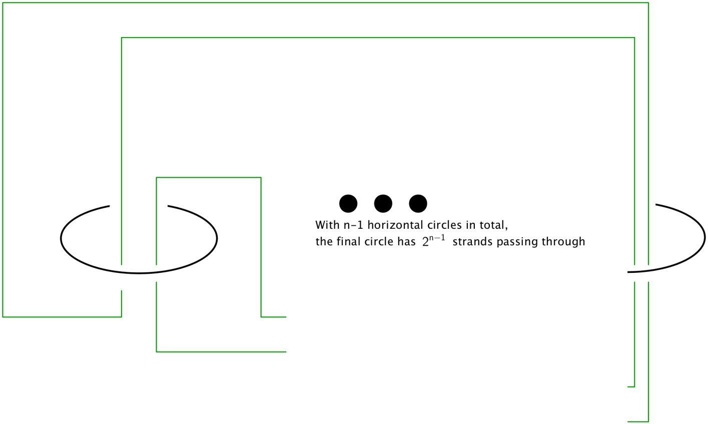
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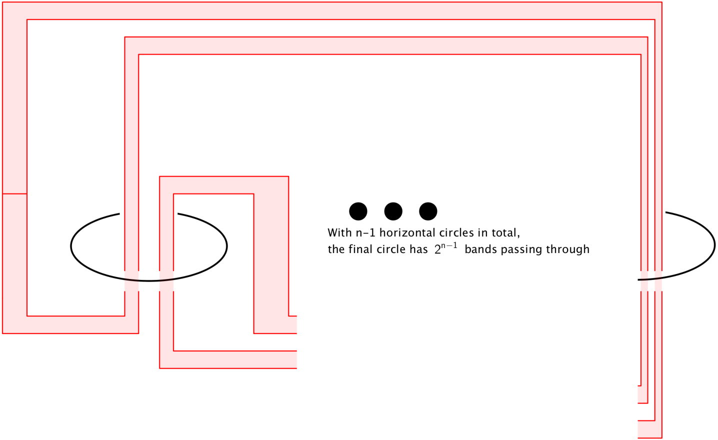
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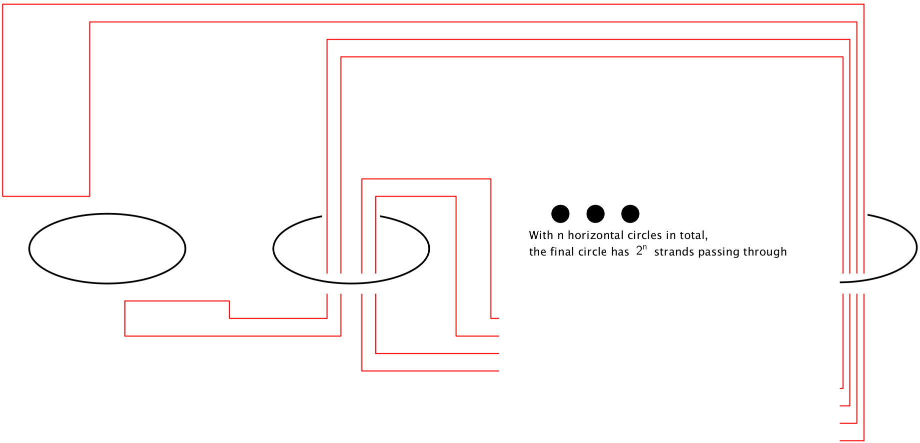
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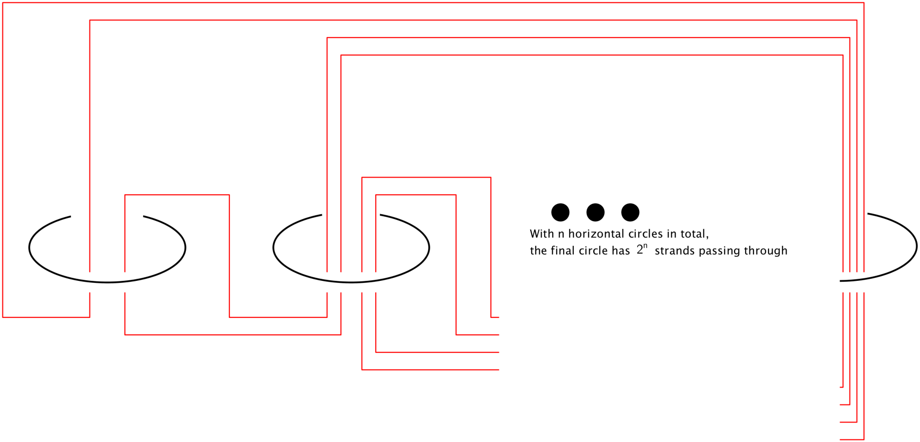
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    - *Trivial k-ary operations for all $k \neq 2, n$*

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- Hopefully our computations in the meantime will confirm the conjecture for $n = 3$

Thank you!