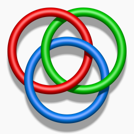
Detecting the Linkage in an *n*-Component Brunnian Link

IMUS Mini-Course Session 2



Joint work with M. Fansler, H. Molina, B. Nimershiem & P. Real

Presented by Dr. Ron Umble

Millersville U and IMUS

2 May 2018

Recap of Lecture 1

- Cell complex X

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- Boundary of a cell

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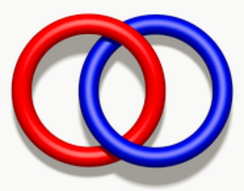
- Geometric diagonal $\Delta_X : X \to X \times X$
- Diagonal (approximation) $\Delta : C(X) \to C(X) \otimes C(X)$
- Δ induces a diagonal $\Delta_2 : H(X) \to H(X) \otimes H(X)$ defined by

$$\Delta_2 [x] = [\Delta (x)]$$

Main result

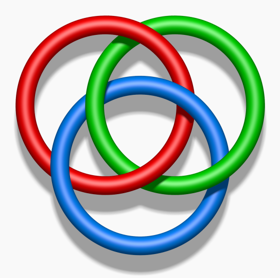
BR_n denote the link complement of an n-component Brunnian Link in S^3

Non-primitivity of the induced Δ_2 *on* $H(BR_2)$ *detects the Hopf link*



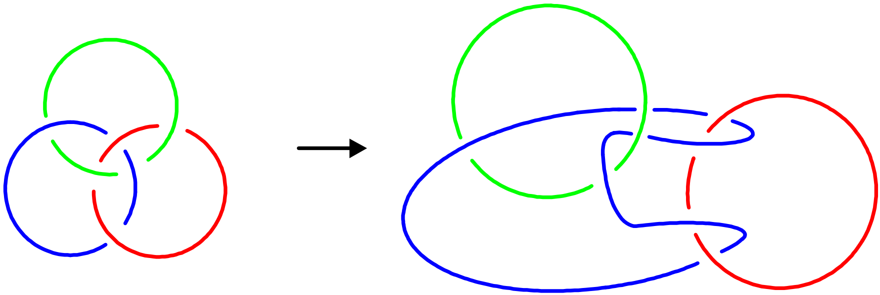
Goal of the Project

Use a similar strategy to detect linkage in an n-component Brunnian link

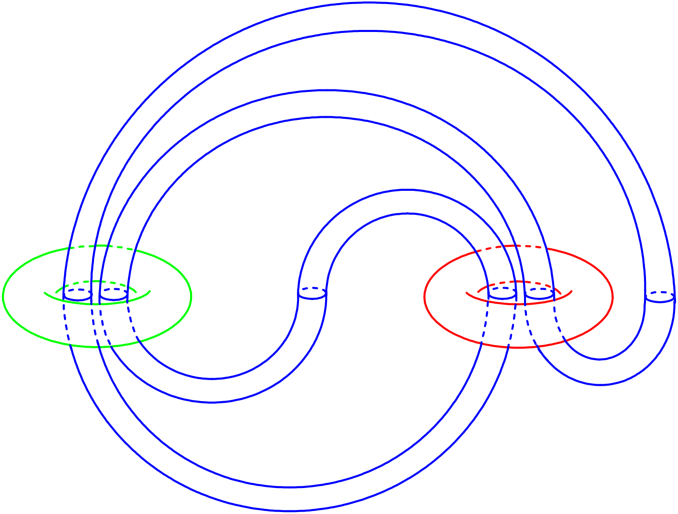


Borromean rings $(n = 3)$

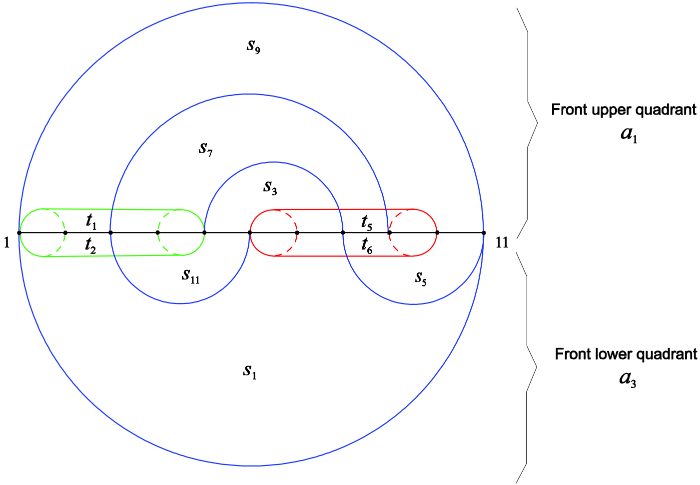
The Borromean Rings



A Tubular Neighborhood of the Borromean Rings



A Cellular Decomposition of BR(3)



2-cells (front side view)

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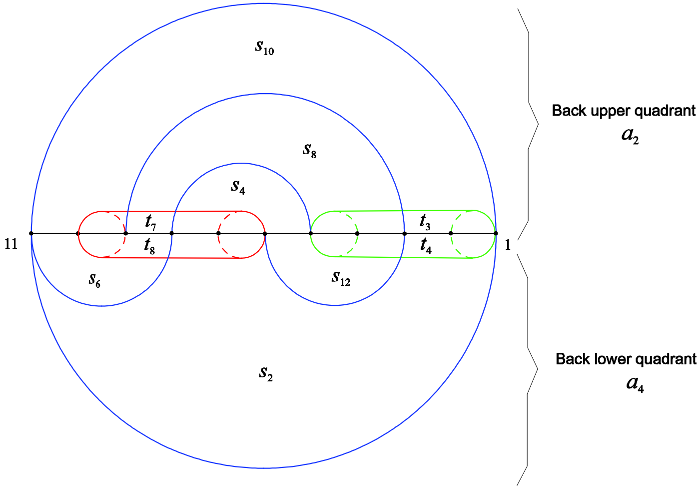

2-cells (back side view)

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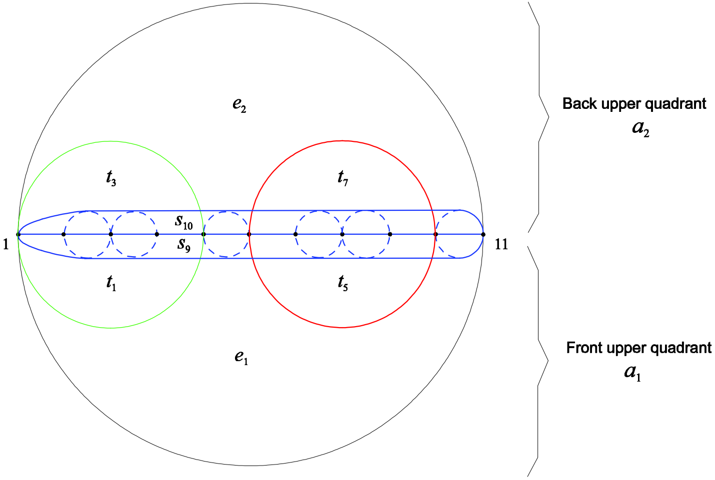

2-cells (top view)

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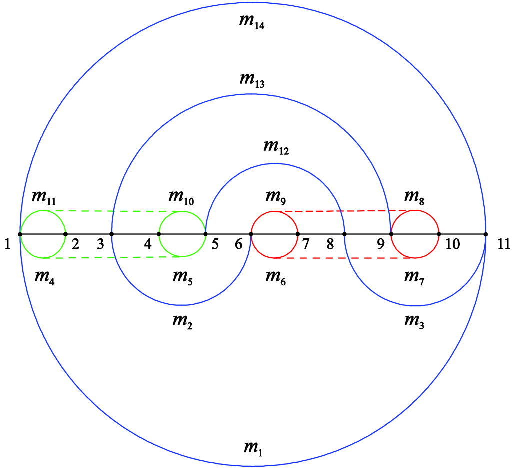

Edges and vertices (front side view)

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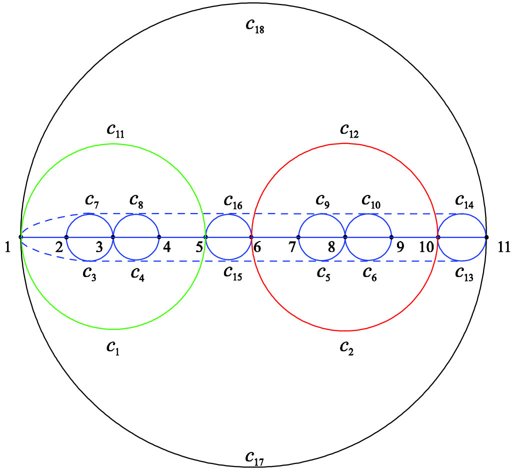

Edges and vertices (top view)

Cellular Chains of BR(3)

- 11 vertices, 32 edges, 26 polygons, 5 solids

$$C_0\left(BR_3\right) = \langle v_1, v_1, \ldots, v_{11} \rangle$$

$$C_1\left(BR_3\right) = \langle m_1, \ldots, m_{14}, c_1, \ldots, c_{18} \rangle$$

$$C_2\left(BR_3\right) = \langle a_1, \ldots, a_4, e_1, e_2, s_1, \ldots, s_{12}, t_1, \ldots, t_8 \rangle$$

$$C_3\left(BR_3\right) = \langle p, q_1, \ldots, q_4 \rangle$$

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- Euler characteristic $11 - 32 + 26 - 5 = 0$

Cellular Boundary Map

- $\partial p = a_1 + a_2 + a_3 + a_4$

  $\partial q_1 = a_1 + e_1 + s_3 + s_7 + s_9 + t_1 + t_5$

  $\partial q_2 = a_2 + e_2 + s_4 + s_8 + s_{10} + t_3 + t_7$

  $\partial q_3 = a_3 + e_1 + s_1 + s_5 + s_{11} + t_2 + t_6$

  $\partial q_4 = a_4 + e_2 + s_2 + s_6 + s_{12} + t_4 + t_8$

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- Boundaries of lower dim'l cells are evident from the pictures

Cellular Homology of BR(3)

- $H_0(BR_3) = \langle [v_1] \rangle$

  $H_1(BR_3) = \langle [m_4 + m_{11}], [m_7 + m_8], [c_{13} + c_{14}] \rangle$

  $H_2(BR_3) = \langle [t_1 + t_2 + t_3 + t_4], [t_5 + t_6 + t_7 + t_8] \rangle$

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- Euler characteristic $1 - 3 + 2 = 0$

The Diagonal on 0 and 1-cells

- Order vertices $v_1 < v_2 < \cdots < v_{11}$ and define

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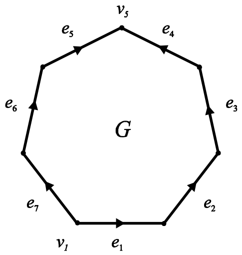
- $\Delta m_i =$(*minimal vertex of* m_i)$\otimes m_i + m_i \otimes$(*maximal vertex of* m_i)

Kravatz's Diagonal on a Polygon

Theorem (Kravatz, 2006) *Let G be an n-gon with initial vertex v_1, terminal vertex v_t, and edges $e_1, e_1, \ldots, e_n$ directed from v_1 to v_t. Then*

$$\Delta(G) = v_1 \otimes G + G \otimes v_t + \sum_{0<i<j<t} e_i \otimes e_j + \sum_{n \geq j > i \geq t} e_j \otimes e_i$$

defines a diagonal on $C(G)$.



A Diagonal on 2-cells

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- $\Delta a_i = v_1 \otimes a_i + a_i \otimes v_{11}$
- $$\Delta t_1 = v_1 \otimes t_1 + t_1 \otimes v_5 + m_{11} \otimes (c_3 + c_4 + m_{10}) + c_3 \otimes (c_4 + m_{10}) + c_4 \otimes m_{10}$$

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- $$\begin{aligned} \Delta q_1 = {} & v_1 \otimes q_1 + q_1 \otimes v_{11} + t_1 \otimes (m_{12} + c_6 + m_8 + c_{13}) \\ & + (c_1 + c_{15}) \otimes t_5 + t_5 \otimes c_{13} \\ & + c_1 \otimes s_3 + s_3 \otimes (c_6 + m_8 + c_{13}) \\ & + (m_{11} + c_3) \otimes s_7 + s_7 \otimes (m_8 + c_{13}) \end{aligned}$$

Conjecture

An extension of Δ *to an* A_∞*-coalgebra structure on* $C(BR_n)$ *induces an* A_∞*-coalgebra structure on* $H(BR_n)$ *with*

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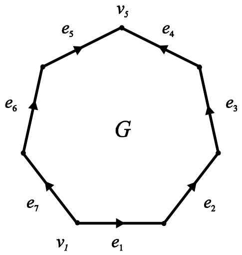
- *Detects the linkage in a n-component Brunnian link*

Minnich's A-infinity Coalgebra Structure on a Polygon

Theorem (Minnich, 2017) *Let G be an n-gon with initial vertex v_1, terminal vertex v_t, and edges $e_1, e_1, \ldots, e_n$ directed from v_1 to v_t. Let Δ_2 denote the Kravatz diagonal. For $k > 2$ define*

$$\Delta_k(G) = \sum_{0<i_1<\cdots<i_k<t} e_{i_1} \otimes \cdots \otimes e_{i_k} + \sum_{n \geq i_1 > \cdots > i_k \geq t} e_{i_1} \otimes \cdots \otimes e_{i_k}.$$

Then $(C(G), \partial, \Delta_2', \Delta_3', \ldots)$ is an A_∞-coalgebra.



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 & & T \searrow & \downarrow^{f+g} & \searrow T & & & \\
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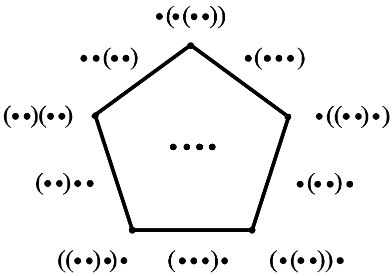
- $f + g$ is a **boundary** if $\delta(T) = f + g$ for some chain homotopy T

Stasheff's Associahedra

- The **associahedron** K_n is an $(n-2)$-dimensional polytope that controls homotopy (co)associativity in n variables



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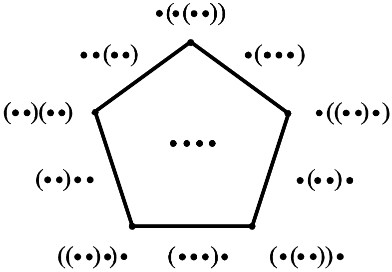

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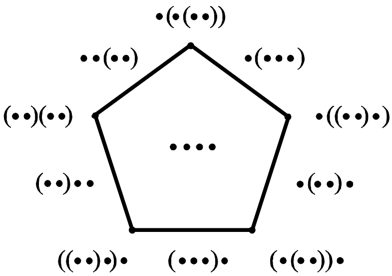

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- For each $n \geq 2$, let θ_n denote the $(n-2)$-dimensional cell of K_n

A-infinity Coalgebras Defined

- *Let* (V, ∂) *be a d.g.v.s. For each* $n \geq 2$, *choose a map* α_n *of deg* 0 :

$$\begin{array}{ccc} C_*(K_n) & \xrightarrow{\alpha_n} & Hom_*(V, V^{\otimes n}) \\ \partial \downarrow & & \downarrow \delta \\ C_{*-1}(K_n) & \underset{\alpha_n}{\longrightarrow} & Hom_{*-1}(V, V^{\otimes n}) \end{array}$$

and define $\Delta_n := \alpha_n(\theta_n)$

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- $(V, \partial, \Delta_2, \Delta_3, \ldots)$ *is an* A_∞-**coalgebra** *if each* α_n *is a chain map, i.e.,*

$$\delta\alpha_n = \alpha_n\partial$$

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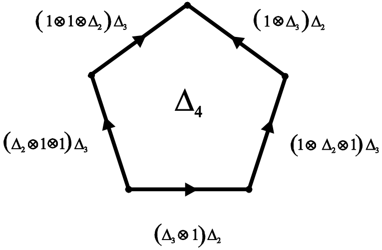
- *Evaluating at* θ_n *produces the classical structure relations*

$$\delta(\Delta_n) = \sum_{i=1}^{n-2}\sum_{j=0}^{n-i-1}(-1)^{i(n+j+1)}\left(\mathbf{1}^{\otimes j}\otimes\Delta_{i+1}\otimes\mathbf{1}^{\otimes n-i-j-1}\right)\Delta_{n-i}$$

Structure Relations

Δ_n is a chain homotopy among the quadratic compositions encoded by the codim 1 cells of K_n



$$\delta\left(\Delta_4\right) = \left(\partial\otimes\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\otimes\partial\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{1}\otimes\partial\right)\Delta_4+\Delta_4\partial$$

$$= \left(\Delta_2\otimes\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\otimes\Delta_2\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{1}\otimes\Delta_2\right)\Delta_3+\left(\Delta_3\otimes\mathbf{1}+\mathbf{1}\otimes\Delta_3\right)\Delta_2$$

Operations on 2-cells of BR(3)

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Transferring Coproducts

Goal:

$$
\begin{array}{c}
A_\infty\text{-coalgebra on chains} \\
(C, \partial, \Delta_2, \Delta_3, \ldots) \\
\downarrow \\
(H, 0, \Delta^2, \Delta^3, \ldots) \\
A_\infty\text{-coalgebra in homology}
\end{array}
$$

Transferring Coproducts

Required input:

- Coalgebra on chains $(C, \partial, \Delta_2, \Delta_3, \ldots)$ and
- a cycle-selecting map $g : H \to Z(C)$, where $Z(C)$ denotes the subspace of cycles in C.

Note: In practice we only required Δ_2 at the outset and computed the rest as needed.

How Does It Work?

Strategy: Construct a chain map from the top dimension and codim-1 cells of the $(n-1)$-dimensional multiplihedron, denoted J_n, to maps between H and $C^{\otimes n}$.

Beginning Steps

- J_n is a polytope that captures the combinatiorial structure of mapping between two A_∞-coalgebras.

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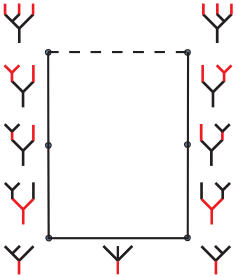
- Consider J_1 and J_2.

Extending to J_3



$\mapsto$

$$g^{\otimes 3}\left(\Delta^2 \otimes \mathbf{1}\right)\Delta^2 \qquad g^{\otimes 3}\left(\mathbf{1} \otimes \Delta^2\right)\Delta^2$$

$$\left(g^2 \otimes g\right)\Delta^2 \qquad \left(g \otimes g^2\right)\Delta^2$$

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Good News

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Bad News

The matrices are too large to be solved within a reasonable amount of storage space and time.

Two Problems

Problem (Preboundary)

Given a cycle $x \in C^{\otimes n}$ of degree k, find a chain $y \in C^{\otimes n}$ of degree $k+1$, such that $\partial(y) = x$.

Problem (Factorization)

Given a cycle $c \in Z(C^{\otimes n})$, find all subcycles of c of the form $Z(C)^{\otimes n}$.

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- Brute force linear algebra approach entails 1.8 mil row $\times$ 4 mil column matrix
- Instead, solved with a best-first search algorithm

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- Again, an algorithmic approach appears to be a feasible alternative

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- Linkage detected but Δ^4 remains to be computed

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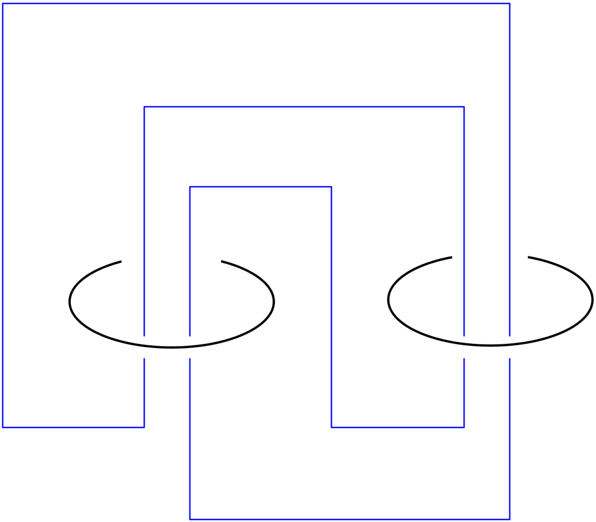
Nimershiem's Decomposition of BR(3)



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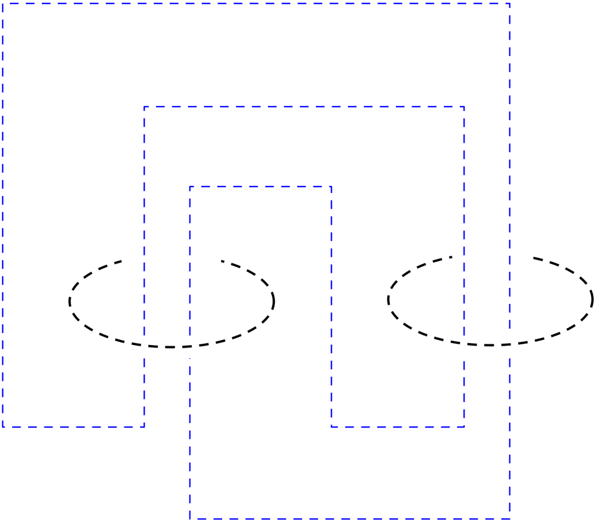

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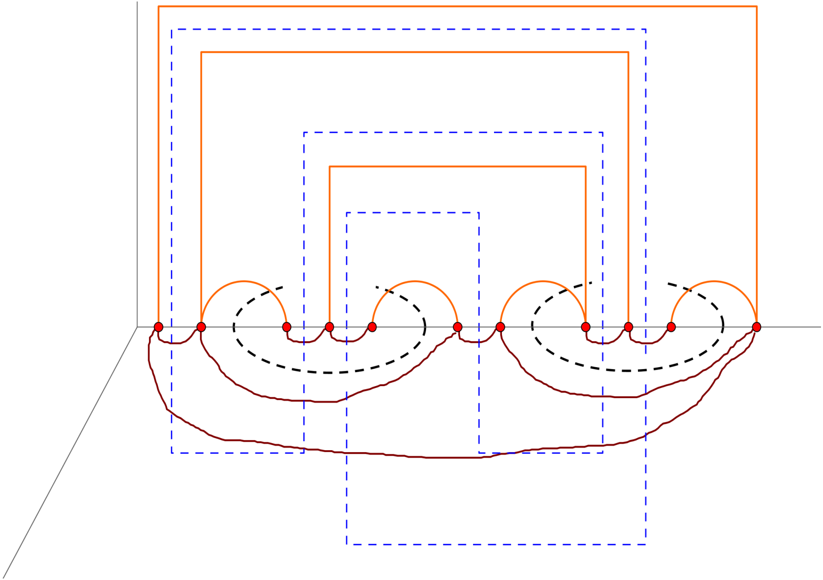

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Thank you!