$A_\infty$-Bialgebras and the Mod 3 Homology of $K(\mathbb{Z},n)$

Presented by

**Ron Umble**

Millersville University of Pennsylvania
ron.umble@millersville.edu

Joint work with

**Ainhoa Berciano and Samson Saneblidze**

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Abstract

A. Berciano applied perturbation methods of T. Kadeishvili and others to define an $A_\infty$-coalgebra structure $\{\Delta_i : H \to H^{\otimes i}\}_{i \geq 2}$ on $H = H_*(\mathbb{Z}, n; \mathbb{Z}_p)$. Since $H$ is a Hopf algebra, $\Delta_2$ is compatible with the Pontryagin product $\mu$ as a map of algebras, and it is natural to ask whether the higher order $\Delta_i$’s are in some sense compatible with $\mu$ as well.

Indeed, let $f = (\Delta_2 \otimes 1) \Delta_2$; when $p = 3$, there is a tensor factor of $H$ of form $E \otimes \Gamma$ on which there are exactly two non-vanishing $A_\infty$-coalgebra operations $\Delta_2$ and $\Delta_3$. We prove that $\Delta_3$ is compatible with $\mu$ as an $(f, f)$-derivation. Thus $E \otimes \Gamma$ is a naturally occurring example of a Hopf $A_\infty$-coalgebra.

Furthermore, $H$ admits operations $\Delta_i$, for all $i \geq 2$. Since $H$ is the homology of a DG Hopf algebra, all $A_\infty$-bialgebra structure relations involving $\Delta_i$ and $\mu$ are satisfied. But whether or not $\Delta_i$ is compatible with $\mu$ as some "higher derivation" is an open question.
**General $A_\infty$-bialgebras**

- Classical constructions:
  
  a. Chain maps

  $$A_{ss} \rightarrow Hom\left(A^{\otimes n}, A\right)$$

  $$A_\infty \rightarrow Hom\left(A^{\otimes n}, A\right)$$

  in the category of non-$\Sigma$ operads define strictly associative and $A_\infty$-algebra structures on $A$

  b. There is a minimal resolution of operads

  $$A_\infty \rightarrow A_{ss}$$

  c. $A_\infty$ is realized by $C_* (K_n)$
• Generalizations:
  
a. \(\text{Ass}\) extends to the bialgebra matrad \(\mathcal{H}\)

b. \(\mathcal{A}_\infty\) extends to \(\mathcal{A}_\infty\)-bialgebra matrad \(\mathcal{H}_\infty\)

c. Chain maps

\[
\mathcal{H} \rightarrow Hom\left(H^\otimes m, H^\otimes n\right)
\]

\[
\mathcal{H}_\infty \rightarrow Hom\left(H^\otimes m, H^\otimes n\right)
\]

in the category if matrads define strictly coassociative and \(\mathcal{A}_\infty\)-bialgebra structures on \(H\)

d. There is a minimal resolution of matrads

\[
\mathcal{H}_\infty \rightarrow \mathcal{H}
\]

e. \(\mathcal{H}_\infty\) is realized by \(C_*\left(KK_{n,m}\right)\)
**Goal:** Specify the combinatorics of $KK_{n,m}$ in dimensions $\leq 3$

- $\dim KK_{n,m} = n + m - 3$

- $KK_{n,1} = KK_{1,n} = K_n$ is Stasheff’s associahedron

**Strategy:** Define cellular boundary $\partial$ on top dim’l faces and extend inductively to lower dimensions.

- $\mathcal{H}_\infty$ has one generator in each bideg $(m, n)$:

  $$
  \theta^m_n = \begin{array}{c}
  \cdots \\
  \cdots \\
  m
  \end{array}
  \begin{array}{c}
  n \\
  \cdots
  \end{array}
  $$

  thought of as an operation in

  $$
  M = \{ M_{n,m} = Hom (H^\otimes m, H^\otimes n) \}
  $$

- Data flows upward
Markl’s Fraction Product on $TM$

Non-zero monomials generated by the $\theta_m^n$’s are “fractions” $\alpha$ of the form

$$\alpha = \frac{\alpha_{xp}^{y_1} \cdots \alpha_{xp}^{y_q}}{\alpha_{x_1}^q \cdots \alpha_{x_p}^q}$$

in which

1. There are $q$ factors above $\Leftrightarrow$ each factor below has $q$ outputs

2. There are $p$ factors below $\Leftrightarrow$ each factor above has $p$ inputs

3. The $j^{th}$ output of the $i^{th}$ factor below links to the $i^{th}$ input of $j^{th}$ factor above

\begin{equation*}
\begin{array}{c}
\text{\begin{tikzpicture}
\node [fill=white] (a) at (0,0) {Y};
\node [fill=white] (b) at (1,0) {Y};
\node [fill=white] (c) at (2,0) {Y};
\node [fill=white] (d) at (3,0) {Y};
\node [fill=white] (e) at (4,0) {Y};
\end{tikzpicture}}
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{c}
\text{\begin{tikzpicture}
\node (a) at (0,0) {\phantom{Y}};
\node (b) at (1,0) {\phantom{Y}};
\node (c) at (2,0) {\phantom{Y}};
\node (d) at (3,0) {\phantom{Y}};
\node (e) at (4,0) {\phantom{Y}};
\end{tikzpicture}}
\end{array}
\end{equation*}

- $\dim \alpha = \sum_{i,j} \dim \alpha_{x_i}^q + \dim \alpha_{xp}^{y_j}$
• The fraction product is not associative:

\[(AB)C \neq 0 = A(BC)\]

• Operad of up-rooted Planar Rooted Trees (PRTs) is the free associative algebra

\[A = \left\langle \ | , \ \ , \ \ , \ \ , \ \ , \ \ldots \right\rangle / \sim\]

• Monomials in A are classes of fraction products

\[
\begin{array}{c}
\text{Tree 1} \\
\text{Tree 2}
\end{array}
\]

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• Differential on $A$:

$$\partial (\theta^1_m) = \sum_{\alpha^1_m \in A \cdot A \atop \dim \alpha^1_m = m-3} \pm \alpha^1_m$$

• Similarly, the operad of down-rooted PRTs

$$C = \langle \mathcal{I}, \mathcal{Y}, \mathcal{Y}, \mathcal{Y}, \ldots \rangle / \sim \text{ with}$$

$$\partial (\theta^n_1) = \sum_{\alpha^n_1 \in C \cdot C \atop \dim \alpha^n_1 = n-3} \pm \alpha^n_1$$

• $A$ and $C$ represent the $A_\infty$-operad $A_\infty$
Matrad Products in dimensions $\leq 3$

- Appropriately restrict the fraction product

Given

$$\alpha = \frac{\alpha_y^1 \cdots \alpha_y^q}{\alpha_x^1 \cdots \alpha_x^q}$$

use its upper and lower leaf sequences

$$y = (y_1, \ldots, y_q) \text{ and } x = (x_1, \ldots, x_p)$$

to decorate $\alpha$:

$$\alpha^y_x = \alpha$$

Example: $\alpha^{32}_{213} =$

\[\begin{array}{ccc}
\big/ & \big/ & \big/ \\
\big/ & \big/ & \\
\big/ & \big/ & \\
\big/ & \big/ & \\
\big/ & \big/ & \\
\big/ & \big/ & \\
\times & \times & \times
\end{array}\]
Use leaf sequences to decorate factors:

\[ \alpha = \frac{\alpha_{p_1}^{y_1} \cdots \alpha_{p_q}^{y_q}}{\alpha_{x_1}^{q_1} \cdots \alpha_{x_p}^{q_p}} \]

The upper and lower contact sequences of \( \alpha \) are

\( (p_1, \ldots, p_q) \) and \( (q_1, \ldots, q_p) \)

**Example:** The upper and lower contact sequences of

\[ \alpha = \frac{\alpha_{p_1} \alpha_{p_2}}{\alpha_{x_1} \alpha_{x_2}} \]

are

\( ((2, 1), (3)) \) and \( ((2), (2), (2)) \).
Diagonal on Associahedra (Saneblidze-U)

Define $\Delta_K : C_* (K_n) \rightarrow C_* (K_n) \otimes C_* (K_n)$ by

$$\Delta_K(\text{ } ) = \text{ } \otimes \text{ } \text{ } \text{ } \text{ }$$

$$\Delta_K(\text{ } ) = \text{ } \otimes \text{ } \text{ } \text{ } \text{ } + \text{ } \otimes \text{ } \text{ } \text{ }$$

$$\Delta_K(\text{ } ) = \text{ } \otimes \text{ } \text{ } \text{ } \text{ } + \text{ } \otimes \text{ } \text{ } \text{ }$$

$$\text{ } \text{ } \text{ } \text{ } \text{ } + \text{ } \otimes \text{ } \text{ } \text{ } \text{ } + \text{ } \otimes \text{ } \text{ } \text{ }$$

$$\text{ } \text{ } \text{ } \text{ } \text{ } + \text{ } \otimes \text{ } \text{ } \text{ } \text{ } + \text{ } \otimes \text{ } \text{ } \text{ }$$

Define the left-iterated diagonal via

$$\Delta_K^{(0)} = \text{id}$$

$$\Delta_K^{(k)} = (\Delta_K \otimes \text{id}^{\otimes k-1}) \Delta_K^{(k-1)}$$
• View each component of $\Delta^{(q-1)}_K(\ldots)$ as a $(p - 3)$-dim’l subcomplex of $K_p \times q$

• A non-zero matrad monomial

$$\alpha = \frac{\alpha_{y_1}^{y_1} \cdots \alpha_{y_q}^{y_q}}{\alpha_{x_1}^{q_1} \cdots \alpha_{x_p}^{q_p}}$$

of dimension $\leq 3$ satisfies

1. UCS $(p_1, \ldots, p_q)$ is the list of LLS’s in some component of $\Delta^{(q-1)}_K(\ldots)$

2. LCS $(q_1, \ldots, q_p)$ is the list of ULS’s of some component of $\Delta^{(p-1)}_K(\ldots)$
Example: In

\[ \alpha = \begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\times \times \times
\end{array} , \]

UCS \((2, 1), (3)\) is the list of LLS’s in

\[ \begin{array}{ccc}
\text{\textcircled{}} & \otimes & \text{\textcircled{}}
\end{array} \text{ in } \Delta^{(1)}_K ( \begin{array}{c}
\text{\textcircled{}}
\end{array} ) \]

LCS \((2), (2), (2)\) is the list of ULS’s in

\[ \begin{array}{ccc}
Y & \otimes & Y \\
\otimes & Y
\end{array} = \Delta^{(2)}_K (Y) \]
Let \( B' = \langle \chi, \Lambda, \Upsilon, \Theta, \Psi, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc \rangle/ \sim \)

For \( m + n \leq 6 \), define

\[
\partial (\theta^m_n) = \sum_{\alpha^m_n \in B \cdot B \atop \dim \alpha^m_n = m + n - 4} \pm \alpha^m_n
\]

Then

\[
\partial (\bigcirc) = \frac{\chi}{\bigcirc} + \frac{\bigcirc \bigcirc \bigcirc}{\Upsilon \Upsilon \Upsilon}
\]

\[
\begin{array}{c}
\bigcirc \\
\hline \\
\bigcirc & \bigcirc \\
\hline \\
\Upsilon \Upsilon \Upsilon & \bigcirc \\
\hline
\end{array}
\]

\[
KK_{2,2}
\]

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\[ \partial(X) = \frac{Y}{1} + \frac{X}{2} + \frac{X}{3} + \frac{X}{4} + \frac{X}{5} + \frac{X}{6} + \frac{X}{7} + \frac{X}{8} \]

\[ KK_{2,3} \]
\[ \partial (\chi) = \frac{Y}{1} + \frac{X}{11} + \frac{X}{111} + \frac{X}{1111} + \frac{X}{11111} \\
+ \frac{X}{1111} + \frac{X}{11111} + \frac{X}{111111} + \frac{X}{1111111} \\
+ \frac{X}{1111111} + \frac{X}{11111111} + \frac{X}{111111111} + \frac{X}{1111111111} \\
+ \frac{X}{1111111111} + \frac{X}{11111111111} + \frac{X}{111111111111} + \frac{X}{1111111111111} \]

\[ KK_{2,4} \]

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\(KK_{3,3}\)
\begin{itemize}
  \item $C_\ast(KK)$ realizes the $A_\infty$-bialgebra matrad $\mathcal{H}_\infty$
  \item $\text{End}(TH)$ is canonically a matrad
  \item A map of matrads
    \[ \mathcal{H}_\infty \to \text{End}(TH) \]
    defines an $A_\infty$-bialgebras structure on $H$.
  \item An $A_\infty$-bialgebra is an algebra over $\mathcal{H}_\infty$
\end{itemize}

Alternatively...
The Biderivative

Monomials

\[
\begin{bmatrix}
\alpha_{x_1}^{y_1} & \cdots & \alpha_{x_p}^{y_1} \\
\vdots & \ddots & \vdots \\
\alpha_{x_1}^{y_q} & \cdots & \alpha_{x_p}^{y_q}
\end{bmatrix}
\in M_x^y
\]

are called bisequence matrices and

\[
M = \bigoplus_{x,y} M_x^y
\]

• There is a product \( \Upsilon : M \times M \to M \)

• For each \( m, n \geq 1 \), choose

\[
\omega_m^n \in Hom\left( H^\otimes m, H^\otimes n \right)
\]

and let \( \omega = \sum \omega_m^n \)

• The “biderivative” \( d_\omega : M \to M \) induces a non-bilinear operation

\[
\odot : M \times M \overset{d_\bullet \times d_\bullet}{\longrightarrow} M \times M \overset{\Upsilon}{\longrightarrow} M \overset{proj}{\longrightarrow} M
\]
Construct $d_\omega$ as follows:

- Linearly extend $d = \omega_1^1$ to $(H^\otimes p)^\otimes q$

- Freely extend the map
  
  $\sum_{j \geq 1} \omega_1^j : H \to T^a H$
  
  as a derivation

- Cofreely extend the map
  
  $\sum_{i \geq 1} \omega_i^1 : T^c H \to H$
  
  as a coderivation

- Freely extend the map
  
  $\sum_{j > 1} \omega_i^j : H^\otimes i \to T^a H$
  
  as a $\Delta_P$-derivation for each $i$

- Cofreely extend the map
  
  $\sum_{i > 1} \omega_i^j : T^c H \to H^\otimes j$
  
  as a $\Delta_P$-coderivation for each $j$
To picture this, make the identification

\[(H \otimes p) \otimes q \leftrightarrow (p, q) \in \mathbb{N}^2\]

and represent \(\omega^q_p : H \otimes p \rightarrow H \otimes q\) as a “transgressive” arrow \((p, 1) \rightarrow (1, q) :\)

- Represent components \(A\) and \(B\) of the extensions above as arrows in \(\mathbb{N}^2\)
• When the terminal point of $B$ is the initial point of $A$ define

$$\gamma(A \otimes B) = A \circ \sigma_{p,q} \circ B$$

where $\sigma_{p,q} : (H^\otimes p)^\otimes q \approx (H^\otimes q)^\otimes p$ is the canonical permutation of tensor factors.

• Define $\omega \odot \omega = \sum_{A,B \in d_\omega} \gamma(A \otimes B)$

• Summands of $\omega \odot \omega$ have one of two types:

1. $\gamma(\omega_j^k \otimes (1 \cdots \omega_i^1 \cdots 1))$ and vise versa

2. $\gamma(A_1 \cdots A_t \otimes B_1 \cdots B_s)$ with $s, t \geq 2$

(sequences of arrows)
• When $\omega \otimes \omega = 0$, there is a relation involving certain “transgressive” products $\gamma(A \otimes B)$ from $(p, 1)$ to $(1, q)$ for each $p$ and $q$

**Example:** When $\omega^2 = 0$ there is the relation

$$d \times + \times d = \times + \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc$$

**Alternative Definition:**

$(H, \omega)$ is an $A_\infty$-bialgebra if $\omega \otimes \omega = 0$
Given a DG $R$-module $(A, \partial)$, $\deg \partial = -1$, $\partial^\otimes n$ denotes the free linear extension to $A^\otimes n$.

For each $n \geq 1$, define

$$\delta^n : \text{Hom} (A, A^\otimes n) \to \text{Hom} (A, A^\otimes n)$$

$$\delta^n (f) = \partial^\otimes n f - (-1)^{|f|} f \partial$$

- $(\text{Hom} (A, A^\otimes \ast), \delta)$ is a DG $R$-module.

Given $\Delta_2 \in \text{Hom}^0 (A, A^\otimes 2)$, let

$f = (\Delta_2 \otimes 1) \Delta_2$ and $g = (1 \otimes \Delta_2) \Delta_2$

- The cochain $g - f \in \text{Hom}^0 (A, A^\otimes 3)$ measures the deviation from coassociativity.
• A coassociating homotopy from \( f \) to \( g \) is a map \( \Delta_3 \in Hom^1(A, A \otimes^3) \) such that

\[
\partial \Delta_3 + \Delta_3 \partial \otimes^3 = g - f
\]

• A chain map \( \xi : A_\infty \to Hom(A, A^{\otimes*}) \) that preserves operadic structure defines an \( A_\infty \)-coalgebra structure on \( A \).

\[
\Delta_2 = \xi (\Upsilon) \text{ is a comultiplication}
\]

\[
\Delta_3 = \xi (\Upsilon) \text{ is a coassociating homotopy}
\]

\[
\Delta_4 = \xi (\Upsilon) \text{ is a homotopy of homotopies}
\]
Theorem (Berciano) For all odd primes $p$,

$$E(v, 2n + 1) \otimes \Gamma(w, 2np + 2)$$

is an $A_\infty$-coalgebra with $\Delta_q \neq 0$ iff $q = 2, p$.

In fact, for $i = 0, 1$ and $\gamma_j = \gamma_j(w)$,

$$\Delta_2(v^i \gamma_j) = \sum_{k=0}^{i} \sum_{l=0}^{j} v^k \gamma_l \otimes v^{i-k} \gamma_{j-l}$$

$$\Delta_p(v^i \gamma_j) = \sum_{k_1 + \cdots + k_p = j-1} v^{i+1} \gamma_{k_1} \otimes \cdots \otimes v^{i+1} \gamma_{k_p}$$

**Question 1:** Since $E(v, 2n+1)\otimes\Gamma(w, 2np+2)$ is a Hopf algebra, $\Delta_2$ is compatible with the multiplication $\mu$ in the sense that $\Delta_2$ is an algebra map. Is $\Delta_p$ in some sense compatible with $\mu$?

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**Δ-Derivations** (Saneblidze, U)

Consider an $A_\infty$-coalgebra $(A, \Delta_n)_{n \geq 2}$ with operadic representation $\xi$ and associative multiplication $\mu : A \otimes A \to A$.

- If $\Delta_2$ is an algebra map, i.e.,

  $\Delta_2 \mu = \mu \otimes^2 \sigma_{2,2} (\Delta_2 \otimes \Delta_2)$

  $= \mu \otimes^2 \sigma_{2,2} [(\xi \otimes \xi) \Delta_K (e^0)]$,

we say that $\Delta_2$ is a $\Delta$-derivation with respect to the empty family $\mathcal{F}_2$ and

$\mathcal{F}_3 = \{ f = (\Delta_2 \otimes 1) \Delta_2, \ g = (1 \otimes \Delta_2) \Delta_2 \} \leftrightarrow \{ \text{faces of } K_3 \text{ with codim } > 0 \}$

is a $\Delta$-compatible family on vertices of $K_3$. 

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• If $\Delta_3$ is a $(f, g)$-derivation, i.e.,

$$
\Delta_3 \mu = \mu \otimes^3 \sigma_{3,2} [f \otimes \Delta_3 + \Delta_3 \otimes g]
$$

$$
= \mu \otimes^2 \sigma_{3,2} [(\xi \otimes \xi) \Delta_K (e^1)],
$$

we say that $\Delta_3$ is a $\Delta$-derivation with respect to $\mathcal{G}_3$ and

$$
\mathcal{G}_4 = \{ \text{compositions involving } \Delta_2, \Delta_3 \} 
$$

$$
\leftrightarrow \{ \text{faces of } K_4 \text{ with codim } > 0 \}
$$

is a $\Delta$-compatible family on the edges and vertices of $K_4$.

Continue inductively:
• If $\Delta_4$ is a $\Delta$-derivation with respect to $\mathcal{F}_4$, i.e., $\Delta_4 \mu = \mu \otimes^4 \sigma_{4,2} \left[ (\xi \otimes \xi) \Delta_K \left( e^2 \right) \right]$,

$\mathcal{F}_5 = \{ \text{compositions involving } \Delta_2, \Delta_3, \Delta_4 \} \leftrightarrow \{ \text{faces of } K_5 \text{ with codim } > 0 \}$

is a $\Delta$-compatible family on the faces, edges and vertices of $K_5$.

**Definition** A Hopf $A_\infty$-coalgebra is an $A_\infty$-coalgebra $(A, \Delta_n)_{n \geq 2}$ with an associative multiplication $\mu$ such that $\Delta_n$ is a $\Delta$-derivation with respect to $\mathcal{F}_n$ for all $n$, i.e.,

$$\Delta_n \mu = \mu \otimes^n \sigma_{n,2} \left[ (\xi \otimes \xi) \Delta_K \left( e^{n-2} \right) \right].$$
Theorem (B-U) For each $i \geq 0$, the $A_\infty$-coalgebra $A_i =$

$$E(v_i, 2n(3^i) + 1) \otimes \Gamma(w_i, 2n(3^{i+1}) + 2)$$

is a Hopf $A_\infty$-coalgebra with operations $\Delta_2, \Delta_3$ and $\mu$.

Proof: With $f = (\Delta_2 \otimes 1) \Delta_2 = (1 \otimes \Delta_2) \Delta_2$, $\Delta_3$ is an $(f, f)$-derivation by the Vandermonde Identity.

Theorem (Eilenberg, Mac Lane) For odd primes $p$,

$$H_* (\mathbb{Z}, n; \mathbb{Z}_p) \approx$$

$$\bigotimes_{i \geq 0} E(v_i, 2np^i + 1) \otimes \Gamma(w_i, 2np^{i+1} + 2)$$

Remark: Extending the $A_\infty$-coalgebra on each factor $E \otimes \Gamma$ to a global structure on $H_* (\mathbb{Z}, n; \mathbb{Z}_p)$ requires tensor products of $A_\infty$-coalgebras.
Tensor Products of $A_\infty$-Coalgebras

**Definition:** (Saneblidze, U) Given $A_\infty$-coalgebras $(A, \xi_A)$ and $(B, \xi_B)$, define $\xi_{A \otimes B}$ as the sum over all $n$ of the compositions

$$C_\ast(K_n) \xrightarrow{\xi_{A \otimes B}} \Hom(A \otimes B, (A \otimes B)^{\otimes n})$$

$$\Delta_K \downarrow \uparrow (\sigma_{n,2})_*$$

$$C_\ast(K_n) \otimes C_\ast(K_n) \xrightarrow{\xi_{A \otimes \xi_B}} \Hom(A \otimes B, A^{\otimes n} \otimes B^{\otimes n})$$

The induced operation of order $n$ is

$$\Delta_n^* = \xi_{A \otimes B} (e^{n-2}) : A \otimes B \rightarrow (A \otimes B)^{\otimes n},$$

where $e^{n-2}$ denotes the top dim’l cell of $K_n$.

**Remark:** The tensor product of $A_\infty$-coalgebras is neither cocommutative nor coassociative.
Let $A = E(v, 2m + 1) \otimes \Gamma (w, 2mp + 2)$ and $B = E'(v', 2n + 1) \otimes \Gamma'(w', 2np + 2)$.

**Theorem (B-U)** The induced operations $\Delta_2^*$, $\Delta_p^*$ and $\Delta_{2p-2}^*$ act non-trivially on $A \otimes B$.

**Theorem (B-U)** At the prime 3, the induced operations $\Delta_i^*$ on $A \otimes B$ vanish for all $i \geq 5$.

**Question 2:** Since the tensor product of Hopf algebras is a Hopf algebra, the operation $\Delta_2^*$ is compatible with the induced multiplication $\mu^*$. Are $\Delta_p^*$ and $\Delta_{2p-2}^*$ in some sense compatible with $\mu^*$?
Proposition 4 (B-U) Let \((A, \mu_A, \Delta, \Delta_3)\) and 
\((B, \mu_B, \psi, \psi_3)\) be Hopf \(A_\infty\)-coalgebras and let

\[
f = (\Delta \otimes 1) \Delta, \quad g = (\psi \otimes 1) \psi,
\]

\[
h = \sigma_{3,2} (f \otimes g) \quad \text{and}
\]

\[
\Delta_3^* = \sigma_{3,2} (f \otimes \psi_3 + \Delta_3 \otimes g).
\]

Then \(\Delta_3^*\) is an \((h, h)\)-derivation.

Proof: Straightforward calculation.

Corollary 5 Let \((A_i, \mu, \Delta, \Delta_3)\) and 
\((A_i', \mu', \psi, \psi_3)\) be Hopf \(A_\infty\)-coalgebras of
form \(E \otimes \Gamma\). Then \((A_i \otimes A_i', \mu^*, \Delta^*, \Delta_3^*)\) is a
Hopf \(A_\infty\)-coalgebra.

Remark: There is an induced operation \(\Delta_4^*\).
Corollary 6 $H = H_*(\mathbb{Z}, n; \mathbb{Z}_3)$ is a Hopf $A_\infty$-coalgebra up to order 3.

Proof: The induced operation $\Delta_3$ on $H \approx \bigotimes_{i \geq 0} A_i$ is the inductive limit of operations $\Delta_{3}^{(k)}$ on $(\bigotimes_{0 \leq i \leq k-1} A_i) \otimes A_k$. 
The Conjecture

Given Hopf $A_\infty$-coalgebras $(A, \mu_A, \Delta, \Delta_3)$ and $(B, \mu_B, \psi, \psi_3)$ let

\[
F_1 = (\Delta \otimes 1 \otimes 1) \Delta_3 \quad G_1 = (\psi \otimes 1 \otimes 1) \psi_3 \\
F_2 = (1 \otimes \Delta \otimes 1) \Delta_3 \quad G_2 = (1 \otimes \psi \otimes 1) \psi_3 \\
F_3 = (1 \otimes 1 \otimes \Delta) \Delta_3 \quad G_3 = (1 \otimes 1 \otimes \psi) \psi_3 \\
F_4 = (\Delta_3 \otimes 1) \Delta \quad G_4 = (\psi_3 \otimes 1) \psi \\
F_5 = (1 \otimes \Delta_3) \Delta \quad G_5 = (1 \otimes \psi_3) \psi
\]

Then Stasheff’s pentagon relation gives

\[
0 = -F_1 + F_2 - F_3 + F_4 + F_5 \\
= -G_1 + G_2 - G_3 + G_4 + G_5
\]

$f = (\Delta \otimes 1 \otimes 1) (\Delta \otimes 1) \Delta$ and $g = (\psi \otimes 1 \otimes 1) (\psi \otimes 1) \psi$ are algebra maps; $F_i$ is an $(f, f)$- and $G_i$ is a $(g, g)$-derivation.

$h = \sigma_{4,2} (f \otimes g)$ is an algebra map;

$H_i = \sigma_{4,2} (f \otimes G_i + F_i \otimes g)$ is a $(h, h)$-derivation and $\mathcal{F}_4 = \{h, H_i\}$ is $\Delta$-compatible.
Let $\xi_A$ and $\xi_B$ be the operadic representation of these $A_\infty$-coalgebra structures. Then

$$\Delta^*_4 = \sigma_{4,2} (\xi_A \otimes \xi_B) \Delta_K (e^2)$$

$$= \sigma_{4,2} (F_2 \otimes G_5 + F_4 \otimes G_2$$

$$-F_1 \otimes G_3 + F_4 \otimes G_5).$$

**Conjecture (B-U)** Let $A = E (v) \otimes \Gamma (w)$ and $B = E (v') \otimes \Gamma (w')$. Then $\Delta^*_4$ is a $\Delta_K$-derivation with respect to $\mathcal{F}_4$, i.e., the 2-dimensional obstruction $\Phi_2$ arising from the non-primitive terms of $\Delta_K (e^2)$ vanishes.

Extend this to $H = H_* (\mathbb{Z}, n; \mathbb{Z}_3)$ inductively. With $k$ tensor factors of form $E \otimes \Gamma$, there is an induced operation $\Delta^*_k + 2$ whose compatibility with $\mu$ as a $\Delta$-derivation is measured by a $k$-dimensional obstruction $\Phi_k$ arising from the non-primitive terms of $\Delta_K (e^k)$. If $\Phi_k = 0$ for all $k$, then $H$ is a Hopf $A_\infty$-coalgebra.