The Coherent Framed Join and Biassociahedra

Joint work with Samson Saneblidze

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2 March 2018

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- However, outside these ranges we are unable to define an operator that simultaneously preserves coherency and satisfies $d^2 = 0$. In fact...
- When $m = n = 4$, Saneblidze constructed an example with the following property: *If we use all available components of the face operator to extend the differential, coherency is lost; if we use only those available components that preserve coherency,* $d^2 \neq 0$

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- Given $\alpha, \beta \in \Omega S$, define the *product* $\alpha \cdot \beta \in \Omega S$ by

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & t \in [0, \frac{1}{2}] \\ \beta(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

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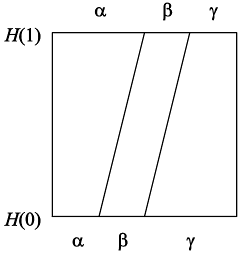
- The loops $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are homotopic via linear change of parameter



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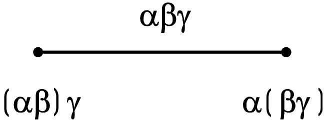

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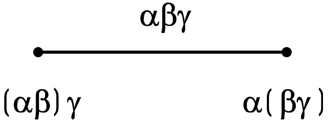
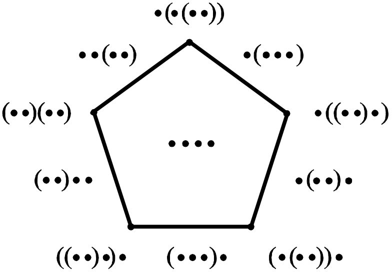


The associahedron K_3

- K_3 controls homotopy associativity in three variables

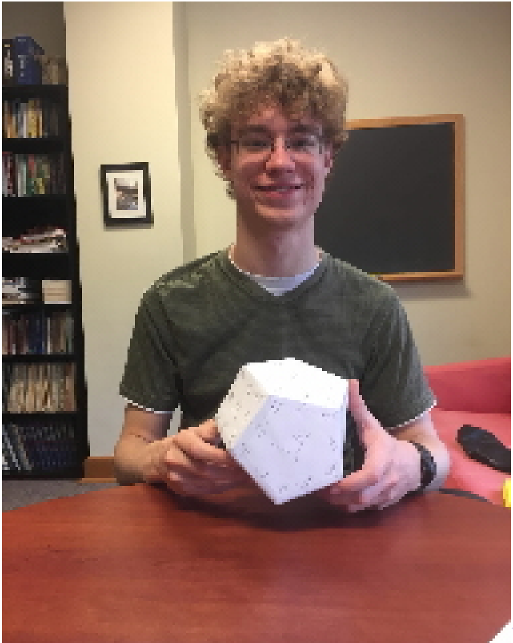
The Associahedron K(4)

- K_4 controls homotopy associativity in four variables



The associahedron K_4

Quinn Minnich's Model of K(5)



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- Let (V, ∂_V) and (W, ∂_W) be d.g.v.s. A linear map $f : V \to W$ has **degree** p if $f : V_i \to W_{i+p}$

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- $\delta(f) = 0$ iff f is a chain map

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- $(V, \partial, \Delta_2, \Delta_3, \ldots)$ is an A_∞-**coalgebra** if for each $n \geq 2$,

$$\delta(\Delta_n) = \alpha_n \partial(\theta^n)$$

Structure Relations

- Expanding $\alpha_n \partial\left(\theta^n\right)$ expresses structure relations in more familiar form:

$$\delta\left(\Delta_n\right) = \sum_{i=1}^{n-2} \sum_{j=0}^{n-i-1} (-1)^{i(n+j+1)} \left(\mathbf{1}^{\otimes j} \otimes \Delta_{i+1} \otimes \mathbf{1}^{\otimes n-i-j-1}\right) \Delta_{n-i}$$

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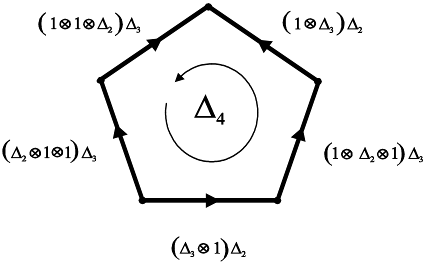
- Δ_n is a chain homotopy among the quadratic compositions encoded by the codim 1 cells of K_n



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- Let us construct $KK_{m,n}$ when $1 \leq m \leq 3$ or $1 \leq n \leq 3$

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- Dimension $\left|A_1|\cdots|A_r\right| := \left|\pi\left(A_1|\cdots|A_r\right)\right|$

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- **Example** $\begin{pmatrix} \frac{4|5}{1|0} & \frac{5|4}{3|2} \\ \frac{7|0|6}{0|1|0} & \frac{67}{23} \end{pmatrix}$ is a bipartition matrix

  over $\mathbf{a}_1 = \{1\}$, $\mathbf{a}_2 = \{2,3\}$, $\mathbf{b}_1 = \{4,5\}$, $\mathbf{b}_2 = \{6,7\}$

  with respect to $\begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$

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- **Example:** $\mu_{\{2,5\}} 2|1|0|5|4|3 = 12|45|3$

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- Let $\bar{A}_i = A_{\lambda^{i-1}+1} \cup \cdots \cup A_{\lambda^i}$
- The λ-**merging map** $\mu_\lambda : P'_{n+1}(A) \to P'_{k+1}(A)$ is defined

$$\mu_\lambda(A_1|\cdots|A_{n+1}) := \bar{A}_1|\cdots|\bar{A}_{k+1}$$

- **Example:** $\mu_{\{2,5\}}2|1|0|5|4|3 = 12|45|3$
- **Extreme cases**: $\mu_\varnothing(A_1|\cdots|A_{n+1}) = A$

$$\mu_{\{1,2,\ldots,n\}}(A_1|\cdots|A_{n+1}) = A_1|\cdots|A_{n+1}$$

Proposition 1

Given a $q \times p$ *bipartition matrix* $\left(\frac{\beta_{ij}}{\alpha_{ij}}\right)$ *over* $\{\mathbf{a}_j, \mathbf{b}_i\}$ *w.r.t.* (r_{ij}), *there is a unique* $q \times p$ *matrix of ordered sets* (λ_{ij}) *such that*

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Example: $\begin{pmatrix} \frac{45|0}{1|0} & \frac{5|4|0}{0|2|3} \\ \frac{7|0|0|6}{0|1|0|0} & \frac{0|7|6}{2|0|3} \end{pmatrix} \overset{\mu_\lambda}{\rightsquigarrow} \begin{pmatrix} \frac{45|0}{1|0} & \frac{45|0}{2|3} \\ \frac{7|6}{1|0} & \frac{7|6}{2|3} \end{pmatrix},$

$$\text{where } \lambda = \begin{pmatrix} \{1\} & \{2\} \\ \{2\} & \{2\} \end{pmatrix}$$

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- **Example**

$$\left(\frac{56|7|8}{1|23|4}\right) = \begin{pmatrix} \frac{56}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{7}{0} & \frac{7}{23} \\ \frac{0}{0} & \frac{0}{23} \end{pmatrix} \begin{pmatrix} \frac{8}{0} & \frac{8}{0} & \frac{8}{0} & \frac{8}{4} \end{pmatrix}$$

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$$\begin{array}{cc} 1 = \mathcal{ACP}_{1} 1 & 56|0|0 = \mathcal{ACP}_{5678} 56 \\ 0|23 = \mathcal{ACP}_{123} 23 & 7|0 = \mathcal{ACP}_{78} 7 \\ 0|0|0|4 = \mathcal{ACP}_{1234} 4 & 8 = \mathcal{ACP}_{8} 8 \end{array}$$

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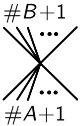
- $\frac{B}{A} \leftarrow$ 

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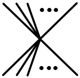
- $\dfrac{B}{A} \leftarrow$ 

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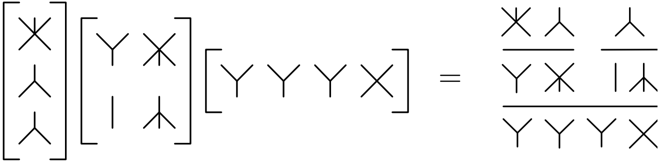
$=$ 

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- Then $|C| := \sum_{1 \leq i \leq q} |\overset{\wedge}{\alpha}_i|$

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- Otherwise...

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- **Example** Discard the 1-dim'l indecomposable matrix

$$C = \begin{pmatrix} \frac{0|1}{1|0} & \frac{0|1}{1|0} & \frac{1}{1} \end{pmatrix}$$

Inserting empty blocks in the third entry transforms C into the 3-dim'l decomposable

$$\begin{pmatrix} \frac{0|1}{1|0} & \frac{0|1}{1|0} & \frac{0|1}{0|1} \end{pmatrix} = \begin{pmatrix} \frac{0}{1} & \frac{0}{1} & \frac{0}{0} \\ \frac{0}{1} & \frac{0}{1} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{1} \end{pmatrix}.$$

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- **Example** Preserve all empty blocks in

$$C = \begin{pmatrix} \frac{0}{1} & \frac{0}{3} \\ \frac{0|0}{1|0} & \frac{0|0}{0|3} \end{pmatrix}$$

Removing empty blocks in the second row increases dimension

Framed Elements

- Given $\mathbf{a}(m)$ and $\mathbf{b}(n)$ of orders m and n, and $r \geq 1$, let

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- Otherwise, assume inductively that the set of framed elements $\alpha' \Cup_f \beta'$ has been defined for all $\frac{\beta'}{\alpha'} \in P'(\mathbf{a}(s)) \times P'(\mathbf{b}(t))$ such that $(s, t) \leq (m, n)$ and $s + t < m + n$

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- The set of **framed elements** $\alpha \Cup_f \beta := \{C_1 \cdots C_r\}$, where C_i ranges over all possible framed matrices and the product is formal juxtaposition

The Framed Join of Ordered Sets

- **Definition** *The* **framed join of a**(m) **and b**(n) *is the set*

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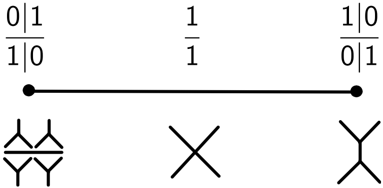

The Hopf relation holds up to homotopy

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- Replace entries in all possible ways to obtain coherence

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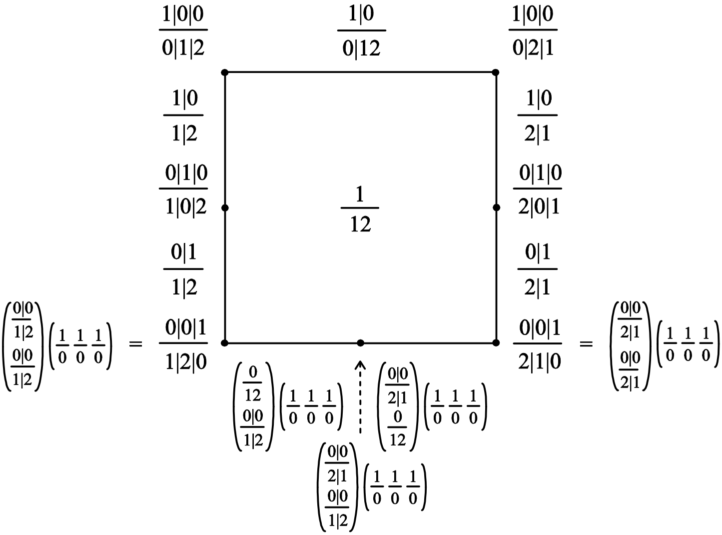
PP(2,3) = KK(2,3)



1|0|0 / 0|1|2
1|0 / 0|12
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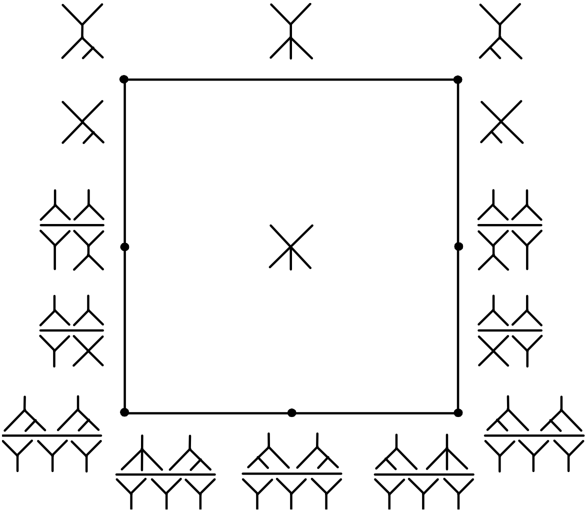

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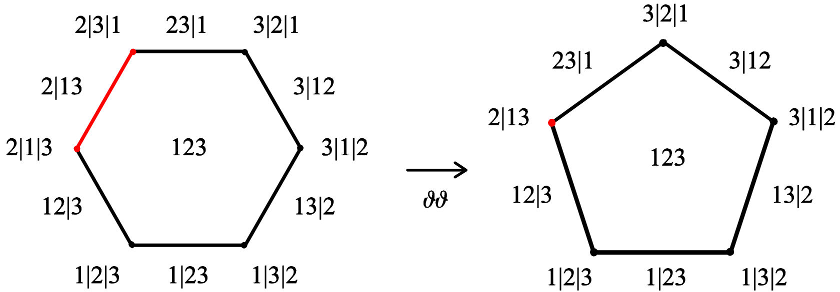
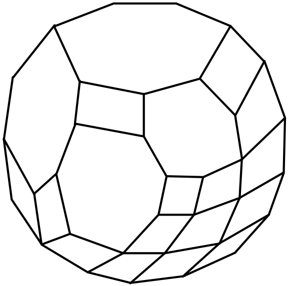
- Canonical projection $\vartheta\vartheta : \mathfrak{m} \circledast_{pp} \mathfrak{n} \to \mathfrak{m} \circledast_{kk} \mathfrak{n}$ is combinatorially equivalent to Tonks' projection when $m = 0$ or $n = 0$

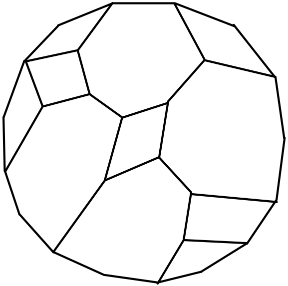
Stasheff's Associahedron K(4)



The Polytope KK(3,3)



Front view



Rear view

- $\partial KK_{3,3}$ consists of 8 heptagons and 22 squares

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- **Definition** *An* A_∞-**bialgebra** *is an algebra over* $\mathcal{H}_\infty$

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- Prior to this work, all known rational homology invariants of $\Omega\Sigma X$ were trivial

The End

Thank you!