Matrads, Matrahedra and $A_\infty$-Bialgebras

Joint work with Samson Saneblidze

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Let $K_n$ denote Stasheff’s $(n-2)$-dim’l associahedron with top dim’l cell $e_n$ and let $K = \sqcup K_n$.
Introduction

- Let $K_n$ denote Stasheff’s $(n-2)$-dim’l associahedron with top dim’l cell $e_n$ and let $K = \sqcup K_n$

- Identify cellular chains $C_\ast(K)$ with the $A_\infty$-operad $A_\infty$
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Identify cellular chains $C_*(K)$ with the $A_\infty$-operad $A_\infty$

An $A_\infty$-algebra is a graded module $A$ together with a family of operations $\{\mu_n \in \text{Hom}(A^\otimes n, A)\}_{n \geq 1}$ and a chain map

$$\varphi : A_\infty \to \{\text{Hom}(A^\otimes n, A)\}_{n \geq 1}$$

such that $\varphi(e_n) = \mu_n$
Let $K_n$ denote Stasheff’s $(n - 2)$-dim’l associahedron with top dim’l cell $e_n$ and let $K = \bigcup K_n$

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In this talk we construct a new family of polyhedra called matrahedra and use them to define the free matrad $\mathcal{H}_\infty$
Let $KK_{t,s}$ denote the $(s + t - 3)$-dim’l matrahedron with top dim’l cell $e_{t,s}$ and let $KK = \sqcup KK_{t,s}$. 

Identify cellular chains $C(CKK)$ with $H \infty A \infty$-bialgebra is a module $H$ together with a family of operations $\theta_{t,s}^2 \in \text{Hom}(H^s, H^t)$ and a chain map $\phi: H \infty ! \text{Hom}(H^s, H^t)$ such that $\phi(e_t, s) = \theta_{t,s}^1$. 

Recently S. Saneblidze proved that if $F$ is a field, $H(\Omega X; F)$ admits a canonical $A \infty$-bialgebra structure $H(\Omega \Sigma X; F)$ is an $A \infty$-bialgebra with operations $\theta_1^2, \theta_{1^n}^2$.
Let $KK_{t,s}$ denote the $(s + t - 3)$-dim’l matrahedron with top dim’l cell $e_{t,s}$ and let $KK = \bigsqcup KK_{t,s}$.

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$H_*(\Omega\Sigma X; F)$ is an $A_\infty$-bialgebra with operations $\{ \theta^1_2, \theta^n_1 \}_{n \geq 2}$
Permutahedra

\[ P_n = \text{convex hull } \{(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n \mid \sigma \in S_n\} \]
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- Vertices $v_1, \ldots, v_n!$ in the hyperplane $x_1 + \cdots + x_n = \binom{n}{2}$
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*Matrads, Matrahedra and $A_\infty$-Bialgebras*
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- $P_1$ is a point *
- $P_2$ is a closed interval $I$
- $P_3$ is a plane hexagonal region
$P_4$ is a solid truncated octahedron
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- $P_n$ is an $(n - 1)$-dim’l polyhedron
Let $n = \{1, 2, \ldots n\}$

- $\{\text{Faces in codim } p\} \leftrightarrow \{\text{Partitions } U_1|\cdots|U_{p+1} \text{ of } n\}$
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- \{Vertices\} $\leftrightarrow S_n = \{$Permutations of $n$\}
Let $n = \{1, 2, \ldots, n\}$

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- $P_1 : * \leftrightarrow \{1\}$
Combinatorics of Permutahedra

Let \( n = \{1, 2, \ldots, n\} \)

- \( \{\text{Faces in codim } p\} \leftrightarrow \{\text{Partitions } U_1 \mid \cdots \mid U_{p+1} \text{ of } n\} \)
- \( \{\text{Vertices}\} \leftrightarrow S_n = \{\text{Permutations of } n\} \)
- \( P_1 : * \leftrightarrow \{1\} \)
- \( P_2 : \text{edge} \leftrightarrow \{12\}; \text{vertices} \leftrightarrow \{1|2, 2|1\} \)
$P_3$ as a subdivision of $P_2 \times I$:

\begin{align*}
3|1|2 & \\
13|2 & \\
1|3|2 & \\
1|23 & \\
1|2|3 &
\end{align*}

\begin{align*}
3|12 & \\
123 & \\
12|3 & \\
2|13 & \\
2|1|3 &
\end{align*}

\begin{align*}
3|2|1 & \\
23|1 & \\
2|3|1 & \\
2|13 & \\
2|1|3 &
\end{align*}
$P_4$ as a subdivision of $P_3 \times I$:
2-faces of $P_4$:
\( \land_n \) and \( \lor_n \) denote the sets of up-rooted and down-rooted planar *binary* trees with \( n \) levels and \( n + 1 \) leaves.

\[
\begin{array}{c}
1 \\
| \\
2 \\
| \\
3 \\
| \\
4
\end{array}
\]

\[ \iff 2 | 4 | 1 | 3 \in \lor_4 \]
\( \wedge_n \) and \( \vee_n \) denote the sets of up-rooted and down-rooted planar *binary* trees with \( n \) levels and \( n + 1 \) leaves.

The bijections \( S_n \leftrightarrow \wedge_n \) and \( S_n \leftrightarrow \vee_n \) transfer the *Bruhat* partial order generated by transpositions

\[
a_1 \mid \cdots \mid a_n < a_1 \mid \cdots \mid a_{i+1} \mid a_i \mid \cdots \mid a_n \text{ iff } a_i < a_{i+1}
\]

to \( \wedge_n \) and \( \vee_n \).
Let $X$ be a polytope; $\mathcal{V}(X)$ denotes the set of vertices of $X$. 
Posets of Vertices

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- Set $PP_{n,0} = PP_{0,n} = P_n$
Posets of Vertices

- Let $X$ be a polytope; $\mathcal{V}(X)$ denotes the set of vertices of $X$
- Set $\mathcal{P}P_{n,0} = \mathcal{PP}_{0,n} = P_n$
- $\mathcal{P}P_{n,0} := \mathcal{V}(\mathcal{PP}_{n,0}) \leftrightarrow \vee_n$
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$PP_{0,n} := \mathcal{V}(PP_{0,n}) \leftrightarrow \land_n$
A. Tonks’ cellular projection $\vartheta : P_n \rightarrow K_{n+1}$ forgets levels
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$\vartheta(a) = \vartheta(b)$ iff $a \cong b$ (as planar trees)
Stasheff’s Associahedra

\[ K_{n+1} = P_n / \sim \]
Stasheff’s Associahedra

- $K_{n+1} = P_n / \sim$
- $K_2 = *$
Stasheff’s Associahedra

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Stasheff’s Associahedra

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- $K_5 :$

J-L Loday’s rendering of $K_5$
Matrahedra

\[ KK_{n+1,1} = P_{n,0}/ \sim \text{ and } KK_{n+1,1} := \mathcal{V}(KK_{n+1,1}) \]
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Matrahedra

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Goal: Construct the matrahedron $KK_{t,s}$ for all $s, t \geq 2$
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- **Goal**: Construct the matrahedron $KK_{t,s}$ for all $s, t \geq 2$

- Our construction has three steps:
Matrahedra

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**Goal:** Construct the matrahedron $KK_{t,s}$ for all $s, t \geq 2$

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  1. **Construct the poset of vertices $PP_{t,s}$**
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**Goal:** Construct the matrahedron $KK_{t,s}$ for all $s, t \geq 2$

Our construction has three steps:

1. **Construct the poset of vertices $\mathcal{P}_t$**
2. **Construct $PP_{t,s}$ as a subdivision of $P_{s+t}$**
Matrahedra

- $KK_{n+1,1} = P_{n,0}/ \sim$ and $KK_{n+1,1} := V(KK_{n+1,1})$
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Goal: Construct the matrahedron $KK_{t,s}$ for all $s, t \geq 2$

Our construction has three steps:

1. Construct the poset of vertices $PP_{t,s}$

2. Construct $PP_{t,s}$ as a subdivision of $P_{s+t}$

3. Form the quotient space $KK_{t+1,s+1} = PP_{t,s}/ \sim$
Markl’s Construction

- $KK_{t,s}$ is identical to M. Markl’s polytope $B^t_s$ in the range $s + t \leq 6$
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- Our construction uses the S-U diagonal on associahedra to control all such choices
Markl’s Construction

- $KK_{t,s}$ is identical to M. Markl’s polytope $B_s^t$ in the range $s + t \leq 6$
- Markl’s construction makes arbitrary choices
- Our construction uses the S-U diagonal on associahedra to control all such choices
- For us, all choices were made once and for all when constructing the S-U diagonal
We are interested in the product posets $\land_t^S$ and $\lor_t^S$ with lexicographic ordering.
Iterated S-U Diagonal

- We are interested in the product posets $\wedge_t^s$ and $\vee_t^s$ with lexicographic ordering.

- $\Delta_P : C_\ast (P_n) \rightarrow C_\ast (P_n) \otimes C_\ast (P_n)$ denotes the S-U diagonal.

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Iterated S-U Diagonal

- We are interested in the product posets $\land_t^{\times s}$ and $\lor_t^{\times s}$ with lexicographic ordering

- $\Delta_P : C_\ast(P_n) \rightarrow C_\ast(P_n) \otimes C_\ast(P_n)$ denotes the S-U diagonal

- Define $\Delta_P^{(0)} = \text{Id}$ and $\Delta_P^{(k)} = \left(\Delta_P \otimes \text{Id}^{\otimes k-1}\right) \Delta_P^{(k-1)}$
Iterated S-U Diagonal

- We are interested in the product posets $\wedge_t^s$ and $\vee_t^s$ with lexicographic ordering.

- $\Delta_P : C_*(P_n) \rightarrow C_*(P_n) \otimes C_*(P_n)$ denotes the S-U diagonal.

- Define $\Delta_P^{(0)} = \text{Id}$ and $\Delta_P^{(k)} = \left( \Delta_P \otimes \text{Id}^{\otimes k-1} \right) \Delta_P^{(k-1)}$.

- $\wedge_n$ and $\gamma^n$ denote the up-rooted and down-rooted $n$-leaf corolla.
Iterated S-U Diagonal

- We are interested in the product posets $\wedge_t^s$ and $\vee_t^s$ with lexicographic ordering.

- $\Delta_P : C_\ast(P_n) \to C_\ast(P_n) \otimes C_\ast(P_n)$ denotes the S-U diagonal.

- Define $\Delta_P^{(0)} = \text{Id}$ and $\Delta_P^{(k)} = \left(\Delta_P \otimes \text{Id}^\otimes k^{-1}\right) \Delta_P^{(k-1)}$.

- $\prec_n$ and $\succ^n$ denote the up-rooted and down-rooted $n$-leaf corolla.

- Think of $\Delta_P^{(t-1)}(\prec_{s+1})$ as an $(s-1)$-dim’l subcomplex of $P_s^\times t$. 

We are interested in the product posets $\wedge_t^{\times s}$ and $\vee_t^{\times s}$ with lexicographic ordering.

$\Delta_P : C_\ast(P_n) \to C_\ast(P_n) \otimes C_\ast(P_n)$ denotes the S-U diagonal.

Define $\Delta_P^{(0)} = \text{Id}$ and $\Delta_P^{(k)} = \left( \Delta_P \otimes \text{Id}^{\otimes k-1} \right) \Delta_P^{(k-1)}$.

$\wedge_n$ and $\vee^n$ denote the up-rooted and down-rooted $n$-leaf corolla.

Think of $\Delta_P^{(t-1)}(\wedge_{s+1})$ as an $(s - 1)$-dim’l subcomplex of $P_s^{\times t}$.

Think of $\Delta_P^{(s-1)}(\vee^{t+1})$ as a $(t - 1)$-dim’l subcomplex of $P_t^{\times s}$. 
We are interested in the product posets $\land_t^{\times s}$ and $\lor_t^{\times s}$ with lexicographic ordering

$\Delta_P : C_{\ast}(P_n) \to C_{\ast}(P_n) \otimes C_{\ast}(P_n)$ denotes the S-U diagonal

Define $\Delta_P^{(0)} = \text{Id}$ and $\Delta_P^{(k)} = \left( \Delta_P \otimes \text{Id}^{\otimes k-1} \right) \Delta_P^{(k-1)}$

$\land_n$ and $\lor^n$ denote the up-rooted and down-rooted $n$-leaf corolla

Think of $\Delta_P^{(t-1)}(\land_{s+1})$ as an $(s - 1)$-dim'l subcomplex of $P_s^{\times t}$

Think of $\Delta_P^{(s-1)}(\lor^{t+1})$ as a $(t - 1)$-dim'l subcomplex of $P_t^{\times s}$

$\Delta_P^{(1)}(\land) = \land \times \land = P_1^{\times 2}$ and $\Delta_P^{(1)}(\lor) = \lor \times \lor = P_1^{\times 2}$

(the index 2 is suppressed)
Iterated S-U Diagonal

\[ \Delta^{(1)}_P (\bigwedge_3) \subset P^{\times 2}_2 \]

\[ \Delta^{(2)}_P (\bigwedge_3) \subset P^{\times 3}_2 \]
Iterated S-U Diagonal

\[ \Delta_p^{(1)} (\Delta_4) \subseteq P_3^{\times 2} \]

\[ \Delta_p^{(2)} (\Delta_4) \subseteq P_3^{\times 3} \]
Key Step

Let $X_s^t = \mathcal{V}\left(\Delta_p^{(t-1)}(\land_{s+1})\right) \subseteq \land_s^\times t$
Key Step

- Let $X^t_s = \mathcal{V} \left( \Delta^{(t-1)}_P (\land s + 1) \right) \subseteq \land^t_s$

- Let $Y^s_t = \mathcal{V} \left( \Delta^{(s-1)}_P (\lor t + 1) \right) \subseteq \lor^s_t$
Key Step

- Let $X_t = \mathcal{V} \left( \Delta_P^{(t-1)} (\land_{s+1}) \right) \subseteq \land_s^t$

- Let $Y_t = \mathcal{V} \left( \Delta_P^{(s-1)} (\lor_{t+1}) \right) \subseteq \lor_t^s$

- $X_t \times Y_t$ is a subposet of $\land_s^t \times \lor_t^s$
An edge of a poset $Q$ is a pair $(u, v) \in Q \times Q$ such that

- $u \leq v$ and
- $u \neq x \neq v$ implies $x = u$ or $x = v$.
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Edges of $X_{s}^{t+1} \times Y_{t}^{s+1}$ represent 1-dim’l elements of $(H_{\infty})_{t,s}$ generated by $\{1, \gamma, \gamma_{3}, \gamma^{3}\}$
An edge of a poset $Q$ is a pair $(u, v) \in Q \times Q$ such that

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Edges of $X^{t+1}_s \times Y^{s+1}_t$ represent 1-dim’l elements of $(\mathcal{H}_\infty)_{t,s}$ generated by $\{1, \gamma, \gamma^3\}$

1-dim’l elements of $(\mathcal{H}_\infty)_{t,s}$ generated by $\{1, \gamma, \gamma^3, X\}$ are represented by edges of a poset $Z_{t,s}$ related to but disjoint from $X^{t+1}_s \times Y^{s+1}_t$
Edges of a Poset

- An edge of a poset $Q$ is a pair $(u, v) \in Q \times Q$ such that
  - $u \leq v$ and
  - $u \leq x \leq v$ implies $x = u$ or $x = v$

- Edges of $X_{s+1}^t \times Y_{t+1}^s$ represent 1-dim'l elements of $(\mathcal{H}_\infty)_{t,s}$ generated by $\{1, \land, \lor, \land_3, \lor^3\}$

- 1-dim'l elements of $(\mathcal{H}_\infty)_{t,s}$ generated by $\{1, \land, \lor, X\}$ are represented by edges of a poset $Z_{t,s}$ related to but disjoint from $X_{s+1}^t \times Y_{t+1}^s$

- $\mathcal{P}\mathcal{P}_{t,s} = X_{s+1}^t \times Y_{t+1}^s \sqcup Z_{t,s}$
Express $x \in X^t_s$ as an $t \times 1$ column matrix of $t$ up-rooted trees.
Vertices as Matrix Products

- Express $x \in X^t_s$ as an $t \times 1$ column matrix of $t$ up-rooted trees

- $X^2_1 = \begin{bmatrix} \text{\_} \\ \text{\_} \\ \text{\_} \end{bmatrix}$
Vertices as Matrix Products

- Express $x \in X_s^t$ as an $t \times 1$ column matrix of $t$ up-rooted trees

- $X^2_1 = \begin{bmatrix} \text{\_} \\ \text{\_} \end{bmatrix}$

- A leveled tree factors uniquely as the composition of its levels

$$\text{\_} = \begin{bmatrix} \text{\_} \\ \text{\_} \end{bmatrix} \begin{bmatrix} \text{\_} & 1 \end{bmatrix}$$
Vertices as Matrix Products

- Express $x \in X_s^t$ as an $t \times 1$ column matrix of $t$ up-rooted trees

- $X_1^2 = \begin{bmatrix} \text{tree} \\ \text{tree} \end{bmatrix}$

- A leveled tree factors uniquely as the composition of its levels

$$\begin{bmatrix} \text{tree} \\ \text{tree} \end{bmatrix} = \begin{bmatrix} \text{tree} \\ \text{tree} \end{bmatrix} \begin{bmatrix} \text{tree} & 1 \end{bmatrix}$$

- Express $x \in X_s^t$ as a formal product $x_1 \cdots x_s$ of $s$ matrices, each with $t$ rows

$$\begin{bmatrix} \text{tree} \\ \text{tree} \\ \text{tree} \end{bmatrix} = \begin{bmatrix} \text{tree} \\ \text{tree} \\ \text{tree} \end{bmatrix} \begin{bmatrix} \text{tree} & 1 \\ \text{tree} \end{bmatrix} \in X_2^2$$
The $t$ rows of $x_i$ are the $i^{th}$ levels of the $t$ trees in $x$.

\[
\begin{bmatrix}
\end{bmatrix} = \begin{bmatrix}
\end{bmatrix} \begin{bmatrix}
\end{bmatrix}
\]
Vertices as Matrix Products

- The $t$ rows of $x_i$ are the $i^{th}$ levels of the $t$ trees in $x$

$$\begin{bmatrix}
\text{Tree A} \\
\text{Tree B} \\
\end{bmatrix} = \begin{bmatrix}
\text{Tree A} \\
\end{bmatrix} \begin{bmatrix}
\text{Tree A} \\
\text{Tree B} \\
\end{bmatrix}$$

- Each row of $x_i$ contains $\exists$ exactly once
Vertices as Matrix Products

- The $t$ rows of $x_i$ are the $i^{th}$ levels of the $t$ trees in $x$
  \[
  \begin{bmatrix}
  \text{\rotatebox{90}{$\ddots$}} \\
  \end{bmatrix} = \begin{bmatrix}
  \text{\rotatebox{90}{$\ddots$}} \\
  \text{\rotatebox{90}{$\ddots$}} \\
  \end{bmatrix} \begin{bmatrix}
  \text{\rotatebox{90}{$\ddots$}} \\
  \end{bmatrix}
  \]

- Each row of $x_i$ contains $\text{\rotatebox{90}{$\ddots$}}$ exactly once

- $X_i^t = \begin{bmatrix}
  \text{\rotatebox{90}{$\ddots$}} \\
  \text{\rotatebox{90}{$\ddots$}} \\
  \text{\rotatebox{90}{$\ddots$}} \\
  \end{bmatrix}$ (t rows)
Vertices as Matrix Products

- The $t$ rows of $x_i$ are the $i^{th}$ levels of the $t$ trees in $x$

\[
\begin{bmatrix}
\uparrow \downarrow \\
\downarrow \uparrow \\
\end{bmatrix} = \begin{bmatrix}
\uparrow \\
\downarrow \\
\end{bmatrix} \begin{bmatrix}
\downarrow \\
\uparrow \\
\end{bmatrix}
\]

- Each row of $x_i$ contains $\uparrow \downarrow$ exactly once

\[
X_1^t = \begin{bmatrix}
\uparrow \\
\vdots \\
\uparrow \\
\end{bmatrix} \quad (t \text{ rows})
\]

- $X_2^2 = \mathcal{V} \left( \Delta^{(1)}_{P} (\uparrow_3) \right) = \left\{ \begin{bmatrix}
\uparrow \\
\uparrow \\
\end{bmatrix} [ \begin{bmatrix}
\uparrow \\
1 \\
\end{bmatrix} ] < \begin{bmatrix}
\uparrow \\
\uparrow \\
\end{bmatrix} [ \begin{bmatrix}
\uparrow \\
1 \\
\end{bmatrix} ] < \begin{bmatrix}
\uparrow \\
\uparrow \\
\end{bmatrix} [ \begin{bmatrix}
1 \\
\uparrow \\
\end{bmatrix} ] \right\}$
Express $y \in Y_t^s$ as a $1 \times s$ row matrix of down-rooted trees.
Vertices as Matrix Products

- Express \( y \in Y_t^s \) as a \( 1 \times s \) row matrix of down-rooted trees
- Express \( y \) as a formal product \( y_t \cdots y_1 \) of \( t \) matrices, each with \( s \) columns
Vertices as Matrix Products

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Vertices as Matrix Products

- Express $y \in Y_t^s$ as a $1 \times s$ row matrix of down-rooted trees
- Express $y$ as a formal product $y_t \cdots y_1$ of $t$ matrices, each with $s$ columns
- The $s$ columns of $y_j$ are the $j^{th}$ levels of the $s$ trees in $y$
- Each column of $y_j$ contains $\gamma$ exactly once
Vertices as Matrix Products

- Express \( y \in Y_t^s \) as a \( 1 \times s \) row matrix of down-rooted trees
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- The \( s \) columns of \( y_j \) are the \( j^{th} \) levels of the \( s \) trees in \( y \)
- Each column of \( y_j \) contains \( \gamma \) exactly once

\[ Y_1^s = [\gamma \cdots \gamma] \quad (s \text{ columns}) \]
Vertices as Matrix Products

- Express $y \in Y_t^s$ as a $1 \times s$ row matrix of down-rooted trees.
- Express $y$ as a formal product $y_t \cdots y_1$ of $t$ matrices, each with $s$ columns.
- The $s$ columns of $y_j$ are the $j^{th}$ levels of the $s$ trees in $y$.
- Each column of $y_j$ contains $\gamma$ exactly once.
- $Y_1^s = [\gamma \cdots \gamma]$ (s columns)
- $Y_2^2 = \mathcal{V} \left( \Delta_p^{(1)}(\gamma^3) \right) = \{ \begin{bmatrix} \gamma & \gamma \\ 1 & 1 \end{bmatrix} [\gamma \gamma] < \begin{bmatrix} \gamma & 1 \\ 1 & \gamma \end{bmatrix} [\gamma \gamma] < \begin{bmatrix} 1 & 1 \\ \gamma & \gamma \end{bmatrix} [\gamma \gamma] \}$
\( X^2_1 \times Y^2_1 = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] [\gamma \gamma] = \wedge^2_1 \times \vee^2_1 \)
Vertices as Matrix Products

- $X_1^2 \times Y_1^2 = \left[ \begin{array}{c} \gamma \\ \gamma \end{array} \right] \left[ \begin{array}{c} \gamma \\ \gamma \end{array} \right] = \wedge_1^2 \times \vee_1^2$

- $X_2^2 \times Y_1^3 = \left\{ u_1 = \left[ \begin{array}{c} \gamma \\ \gamma \\ \gamma \end{array} \right] \left[ \begin{array}{c} \gamma \\ \gamma \\ 1 \end{array} \right] \left[ \begin{array}{c} \gamma \\ \gamma \\ \gamma \end{array} \right] < u_2 = \left[ \begin{array}{c} \gamma \\ \gamma \\ \gamma \end{array} \right] \left[ \begin{array}{c} \gamma \\ \gamma \\ 1 \end{array} \right] \left[ \begin{array}{c} \gamma \\ \gamma \\ \gamma \end{array} \right] < u_3 = \left[ \begin{array}{c} \gamma \\ \gamma \\ \gamma \end{array} \right] \left[ \begin{array}{c} 1 \\ \gamma \\ \gamma \end{array} \right] \left[ \begin{array}{c} \gamma \\ \gamma \\ \gamma \end{array} \right] \right\} \subset \wedge_2^2 \times \vee_1^3$
Consider a bigraded module $M = \{ M_{q,p} \}_{q,p \geq 1}$.
Consider a bigraded module $M = \{ M_{q,p} \}_{q,p \geq 1}$

**Main Example:**

Let $H$ be a free module of finite type and let

$$ M = \text{End} ( TH ) = \{ M_{q,p} = \text{Hom} ( H^\otimes p, H^\otimes q ) \} $$
Some Algebra – Submodules of TTM

- Consider a bigraded module $M = \{ M_{q,p} \}_{q,p \geq 1}$

- **Main Example:**

  Let $H$ be a free module of finite type and let

  $$M = \text{End} (TH) = \{ M_{q,p} = \text{Hom} (H^\otimes p, H^\otimes q) \}$$

- Think of $\alpha^q_p \in M_{q,p}$ as a multilinear operation $H^\otimes p \to H^\otimes q$
Consider a bigraded module $M = \{M_{q,p}\}_{q,p \geq 1}$

**Main Example:**

Let $H$ be a free module of finite type and let

$$M = \text{End} \left( TH \right) = \left\{ M_{q,p} = \text{Hom} \left( H^{\otimes p}, H^{\otimes q} \right) \right\}$$

Think of $\alpha^q_p \in M_{q,p}$ as a multilinear operation $H^{\otimes p} \rightarrow H^{\otimes q}$

Pictured as a non-planar upward-directed graph

\[ \in M_{2,3} \]
The Matrix Submodule of TTM

- The *matrix* submodule of TTM is

\[
\overline{M} = \bigoplus_{p,q \geq 1} (M^\otimes p)^\otimes q
\]
The Matrix Submodule of TTM

- The *matrix* submodule of $TTM$ is

$$\overline{M} = \bigoplus_{p,q \geq 1} (M^\otimes p)^\otimes q$$

- A pair of matrices $X = (x_{ij})$, $Y = (y_{ij}) \in \mathbb{N}^{q \times p}$ uniquely determines a submodule

$$\overline{M}_X^Y = \begin{pmatrix}
(M_{y_{11}}, x_{11}) \otimes \cdots \otimes (M_{y_{1p}}, x_{1p}) \\
(M_{y_{21}}, x_{21}) \otimes \cdots \otimes (M_{y_{2p}}, x_{2p}) \\
\vdots \\
(M_{y_{q1}}, x_{q1}) \otimes \cdots \otimes (M_{y_{qp}}, x_{qp})
\end{pmatrix} \subset \overline{M}$$
The Matrix Submodule of TTM

- The *matrix* submodule of $TTM$ is

$$\mathcal{M} = \bigoplus_{p, q \geq 1} (M^\otimes p)^\otimes q$$

- A pair of matrices $X = (x_{ij}), \ Y = (y_{ij}) \in \mathbb{N}^{q \times p}$ uniquely determines a submodule

$$\mathcal{M}_Y^X = (M_{y_{11}, x_{11}} \otimes \cdots \otimes M_{y_{1p}, x_{1p}}) \otimes (M_{y_{21}, x_{21}} \otimes \cdots \otimes M_{y_{2p}, x_{2p}}) \otimes \cdots \otimes (M_{y_{q1}, x_{q1}} \otimes \cdots \otimes M_{y_{qp}, x_{qp}}) \subset \mathcal{M}$$

- Each of the $q$ rows is a submodule of $M^\otimes p$
Matrix Monomials in TTM

- $G \subset M$ is a fixed set of bihomogeneous module generators
Matrix Monomials in TTM

- $G \subset M$ is a fixed set of bihomogeneous module generators
- A monomial in $TM$ is an element $A \in G^\otimes p$
\begin{itemize}
  \item $G \subset M$ is a fixed set of bihomogeneous module generators
  \item A monomial in $TM$ is an element $A \in G^{\otimes p}$
  \item A monomial in $TTM$ is an element $B \in (G^{\otimes p})^{\otimes q}$
\end{itemize}
- $G \subseteq M$ is a fixed set of bihomogeneous module generators

- A *monomial in TM* is an element $A \in G^{\otimes p}$

- A *monomial in TTM* is an element $B \in (G^{\otimes p})^{\otimes q}$

- $B$ is represented by the $q \times p$ matrix

$$[B] = \begin{bmatrix}
g^{y_{1,1}}_{x_{1,1}} & \cdots & g^{y_{1,p}}_{x_{1,p}} \\
g^{y_{q,1}}_{x_{q,1}} & \cdots & g^{y_{q,p}}_{x_{q,p}}
\end{bmatrix}$$

with entries in $G$ and rows identified with elements of $G^{\otimes p}$
The component

\[(M \otimes p) \otimes q = \bigoplus_{X, Y \in \mathbb{N}^{q \times p}} \overline{M}^Y_X\]
The component

\[(\mathcal{M} \otimes \rho) \otimes g = \bigoplus_{X,Y \in \mathbb{N}^{q \times p}} \overline{\mathcal{M}}_X^Y\]

Thus

\[\overline{\mathcal{M}} = \bigoplus_{X,Y \in \mathbb{N}^{q \times p}, \rho, q \geq 1} \overline{\mathcal{M}}_X^Y\]
The Bisequence Submodule of TTM

- $X \in \mathbb{N}^{q \times p}$ be a matrix with constant columns; let $\mathbf{x} \in \mathbb{N}^{1 \times p}$ denote any row of $X$
The Bisequence Submodule of TTM

- $X \in \mathbb{N}^{q \times p}$ be a matrix with constant columns; let $\mathbf{x} \in \mathbb{N}^{1 \times p}$ denote any row of $X$

- $Y \in \mathbb{N}^{q \times p}$ be a matrix with constant rows; let $\mathbf{y} \in \mathbb{N}^{q \times 1}$ denote any column of $Y$
The Bisequence Submodule of TTM

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- Let $M_x^y = \overline{M}_x^y$
The Bisequence Submodule of TTM

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- Let $M^y_x = \overline{M}^y_X$

- The bisequence submodule of $TTM$ is

$$M = \bigoplus_{x,y \in \mathbb{N}^{q \times p} \atop p, q \geq 1} M^y_x$$
The Bisequence Submodule of TTM

- $X \in \mathbb{N}^{q \times p}$ be a matrix with constant columns; let $\mathbf{x} \in \mathbb{N}^{1 \times p}$ denote any row of $X$

- $Y \in \mathbb{N}^{q \times p}$ be a matrix with constant rows; let $\mathbf{y} \in \mathbb{N}^{q \times 1}$ denote any column of $Y$

- Let $M^Y_X = \overline{M}_X$

- The bisequence submodule of $TTM$ is

\[
M = \bigoplus_{\mathbf{x}, \mathbf{y} \in \mathbb{N}^{q \times p}} M^Y_X
\]

\[
\text{subject to } p, q \geq 1
\]

- A monomial $A \in M$ is represented by a bisequence matrix

\[
A = \begin{bmatrix}
\alpha_{x1}^y & \cdots & \alpha_{xp}^y \\
\vdots & \ddots & \vdots \\
\alpha_{x1}^q & \cdots & \alpha_{xp}^q
\end{bmatrix}
\]
We refer to $A$ as a $q \times p$ monomial
Bisequence Matrices

- We refer to $A$ as a $q \times p$ monomial

\[
\begin{align*}
A & \in \mathbf{M}_{23} \\
\frac{2}{4} & \\
\end{align*}
\]

- A **Transverse Pair** (TP) of bisequence matrices has form

\[
\begin{bmatrix}
\alpha_p^{y_1} \\
\vdots \\
\alpha_p^{y_q}
\end{bmatrix} \otimes \begin{bmatrix}
\beta_{x_1}^q \\
\cdots \\
\beta_{x_p}^q
\end{bmatrix} \in \mathbf{M}_p^y \otimes \mathbf{M}_x^q
\]
$A \otimes B \in \overline{\mathbb{M}} \otimes \overline{\mathbb{M}}$ is a **Block Transverse Pair** (BTP) if there exist block decompositions $A = [A_{ij}]$ and $B = [B_{ij}]$ such that $A_{ij} \otimes B_{ij}$ is a TP for each $(i, j)$.
- $A \otimes B \in \mathcal{M} \otimes \mathcal{M}$ is a **Block Transverse Pair** (BTP) if there exist block decompositions $A = [A_{ij}]$ and $B = [B_{ij}]$ such that $A_{ij} \otimes B_{ij}$ is a TP for each $(i, j)$.

- An element $A \otimes B \in \mathcal{M}_{21} \otimes \mathcal{M}_{123}$ is a BTP via

$$A = \begin{bmatrix}
\alpha_1^1 \\
\alpha_2^1 \\
\alpha_2^2 \\
\alpha_2^3 \\
\alpha_2^4 \\
\alpha_2^5 \\
\alpha_1^1 \\
\alpha_1^4 \\
\alpha_1^3 \\
\alpha_1^2
\end{bmatrix} \quad B = \begin{bmatrix}
\beta_1^3 & \beta_2^3 & \beta_3^3 \\
\beta_1^1 & \beta_2^1 & \beta_3^1 \\
\beta_1^1 & \beta_2^2 & \beta_3^2 \\
\beta_1^2 & \beta_2^1 & \beta_3^1
\end{bmatrix}$$
A \otimes B \in \mathbf{M} \otimes \mathbf{M} is a **Block Transverse Pair** (BTP) if there exist block decompositions \( A = [A_{ij}] \) and \( B = [B_{ij}] \) such that \( A_{ij} \otimes B_{ij} \) is a TP for each \((i,j)\)

An element \( A \otimes B \in \mathbf{M}_{21} \otimes \mathbf{M}_{123} \) is a BTP via

\[
A = \begin{bmatrix}
\alpha_{12} \alpha_{11} \\
\alpha_{22} \alpha_{21} \\
\alpha_{32} \alpha_{31}
\end{bmatrix}, \quad  B = \begin{bmatrix}
\beta_{13} \beta_{12} \beta_{11} \\
\beta_{23} \beta_{22} \beta_{21}
\end{bmatrix}
\]

BTP decomposition is unique when it exists
Given a map on TPs

\[ \gamma = \left\{ \gamma_x^y : M_p^y \otimes M_q^x \rightarrow M_{\left| y \right| x} \right\}, \]

extend to a global product \( \Upsilon : \overline{M} \otimes \overline{M} \rightarrow \overline{M} \) by setting \( \Upsilon (A \otimes B) = 0 \) unless \( A \otimes B \) is a BTP, in which case define

\[ \Upsilon (A \otimes B)_{ij} = \gamma (A_{ij} \otimes B_{ij}) \]
Given a map on TPs

$$\gamma = \left\{ \gamma^y_x : \mathbb{M}^y_p \otimes \mathbb{M}^q_x \to \mathbb{M} \right\},$$

extend to a global product $\Upsilon : \overline{\mathbb{M}} \otimes \overline{\mathbb{M}} \to \overline{\mathbb{M}}$ by setting $\Upsilon (A \otimes B) = 0$ unless $A \otimes B$ is a BTP, in which case define

$$\Upsilon (A \otimes B)_{ij} = \gamma (A_{ij} \otimes B_{ij})$$

$x_1 \cdot \cdot \cdot x_s \in X^t_s$ and $y_t \cdot \cdot \cdot y_1 \in Y^s_t$ are $\Upsilon$-products
Given a map on TPs

\[ \gamma = \left\{ \gamma_x^y : \mathbf{M}_p^y \otimes \mathbf{M}_x^q \to \mathbf{M}_{x^y} \right\}, \]

extend to a global product \( \Upsilon : \overline{\mathbf{M}} \otimes \overline{\mathbf{M}} \to \overline{\mathbf{M}} \) by setting

\[ \Upsilon(A \otimes B) = 0 \] unless \( A \otimes B \) is a BTP, in which case define

\[ \Upsilon(A \otimes B)_{ij} = \gamma(A_{ij} \otimes B_{ij}) \]

\( x_1 \cdots x_s \in X_s^t \) and \( y_t \cdots y_1 \in Y_t^s \) are \( \Upsilon \)-products

There is unique \( \Upsilon \)-factorization in \( \overline{\mathbf{M}} \)
Upsilon Products

\[\Upsilon : \mathbf{M}_{21}^3 \otimes \mathbf{M}_{123}^1 \rightarrow \mathbf{M}_{33}^3\]

\[
\begin{bmatrix}
\alpha_2^1 & \alpha_1^1 \\
\alpha_2^5 & \alpha_1^5 \\
\alpha_2^4 & \alpha_1^4 \\
\alpha_2^3 & \alpha_1^3 \\
\end{bmatrix} \quad \begin{bmatrix}
\beta_1^3 & \beta_2^3 & \beta_3^3 \\
\beta_1^1 & \beta_2^1 & \beta_3^1 \\
\end{bmatrix} = 
\begin{bmatrix}
\alpha_1^1 \\
\alpha_1^5 \\
\alpha_1^4 \\
\alpha_1^3 \\
\end{bmatrix} \quad \begin{bmatrix}
\beta_1^3 & \beta_2^3 & \beta_3^3 \\
\beta_1^1 & \beta_2^1 & \beta_3^1 \\
\end{bmatrix} \quad \begin{bmatrix}
\alpha_1^1 \\
\alpha_1^5 \\
\alpha_1^4 \\
\alpha_1^3 \\
\end{bmatrix}
\]
When $\alpha^j_i$ is thought of as an element of $\text{Hom}(H^\otimes i, H^\otimes j)$ we can picture $\Upsilon$-products graphically.
When $\alpha_j^i$ is thought of as an element of $\text{Hom}(H^\otimes i, H^\otimes j)$ we can picture $\Upsilon$-products graphically.

\[
\begin{array}{c}
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\begin{bmatrix}
\gamma \\
\gamma
\end{bmatrix} =
\begin{array}{c}
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\begin{bmatrix}
\gamma \\
\gamma
\end{bmatrix} =
\end{array}
\end{array}
\]
When $\alpha^j_i$ is thought of as an element of $\text{Hom}(H^\otimes i, H^\otimes j)$ we can picture $\Upsilon$-products graphically:

$$\left[ \begin{array}{c} \uparrow \\ \uparrow \end{array} \right] \left[ \begin{array}{c} \uparrow 1 \\ \downarrow \end{array} \right] [\Upsilon \Upsilon \Upsilon] = \frac{\ \ \ \ }{\ \ \ \ } = \frac{\ \ \ \ }{\ \ \ \ }$$

The picture in the center is M. Markl’s *fraction product*.
A pair or matrices \((A, B)\) is an \((i, j)\)-edge pair if

- \(A = [a_{ij}]\) is a matrix over \(\{1, \wedge\}\) whose rows contain \(\wedge\) exactly once
- \(B = [b_{ij}]\) is a matrix over \(\{1, \vee\}\) whose col’s contain \(\vee\) exactly once
- \[
\begin{bmatrix}
    a_{ij} \\
    a_{i+1,j}
\end{bmatrix}
\begin{bmatrix}
    b_{ij} & b_{i,j+1}
\end{bmatrix} =
\begin{bmatrix}
    \wedge \\
    \wedge
\end{bmatrix}
\begin{bmatrix}
    \vee & \vee
\end{bmatrix}
\]
A pair or matrices \((A, B)\) is an \((i, j)\)-edge pair if

- \(A = [a_{ij}]\) is a matrix over \(\{1, \wedge\}\) whose rows contain \(\wedge\) exactly once
- \(B = [b_{ij}]\) is a matrix over \(\{1, \gamma\}\) whose col’s contain \(\gamma\) exactly once

\[
\begin{bmatrix}
a_{ij} \\
a_{i+1,j}
\end{bmatrix}
\begin{bmatrix}
b_{ij} & b_{i,j+1}
\end{bmatrix}
= 
\begin{bmatrix}
\wedge \\
\wedge
\end{bmatrix}
\begin{bmatrix}
\gamma & \gamma
\end{bmatrix}
\]

\((A_s, B_t)\) is the only possible edge pair of adjacent matrices in

\[
A_1 \cdots A_s B_t \cdots B_1 \in X_{s}^{t+1} \times Y_{t}^{s+1}
\]
(\(x_2, y_2\)) is a \((1, 1)\)-edge pair in

\[
x_1x_2y_2y_1 = \begin{bmatrix}
\otimes \\
\otimes \\
\otimes
\end{bmatrix} \begin{bmatrix}
\otimes \\
\otimes \\
\otimes
\end{bmatrix} \begin{bmatrix}
\otimes \\
\otimes \\
\otimes
\end{bmatrix} \begin{bmatrix}
\otimes \\
\otimes \\
\otimes
\end{bmatrix}
\]

There are no adjacent edge pairs in

\[
\begin{bmatrix}
\otimes \\
\otimes \\
\otimes
\end{bmatrix} \begin{bmatrix}
\otimes \\
\otimes \\
\otimes
\end{bmatrix} \begin{bmatrix}
\otimes \\
\otimes \\
\otimes
\end{bmatrix}
\]
(x_2, y_2) is a (1,1)-edge pair in

\[ x_1 x_2 y_2 y_1 = \begin{bmatrix} 1 & 1 \\ \end{bmatrix} \begin{bmatrix} 1 & \gamma & 1 \\ \end{bmatrix} [\gamma \gamma \gamma] \]

There are no adjacent edge pairs in

\[ \begin{bmatrix} 1 & 1 \\ \end{bmatrix} \begin{bmatrix} 1 & \gamma & \gamma \\ \end{bmatrix} [\gamma \gamma \gamma] \]
Matrix Transpositions

- $A^{i*}$ denotes the matrix obtained by deleting the $i^{th}$ row of $A$
Matrix Transpositions

- $A^i$ denotes the matrix obtained by deleting the $i^{th}$ row of $A$
- $B^j$ denotes the matrices obtained by deleting the $j^{th}$ column of $B$
Matrix Transpositions

- $A^i*$ denotes the matrix obtained by deleting the $i^{th}$ row of $A$
- $B^j$ denotes the matrices obtained by deleting the $j^{th}$ column of $B$
- If $(A, B)$ is an $(i, j)$-edge pair, the $(i, j)$-transposition of $AB$ is $B^j A^i*$
Matrix Transpositions

- $A^{i*}$ denotes the matrix obtained by deleting the $i^{th}$ row of $A$
- $B^{*j}$ denotes the matrices obtained by deleting the $j^{th}$ column of $B$
- If $(A, B)$ is an $(i, j)$-edge pair, the $(i, j)$-transposition of $AB$ is $B^{*j}A^{i*}$
- The $(1, 1)$-transposition of $\begin{pmatrix} \& 1 \\ \& 1 \end{pmatrix} [\gamma \gamma \gamma]$ is $[\gamma \gamma] \begin{pmatrix} \& 1 \\ \& 1 \end{pmatrix}$
Matrix Transpositions

- \( A^{i*} \) denotes the matrix obtained by deleting the \( i^{th} \) row of \( A \)
- \( B^{*j} \) denotes the matrices obtained by deleting the \( j^{th} \) column of \( B \)
- If \((A, B)\) is an \((i, j)\)-edge pair, the \((i, j)\)-transposition of \(AB\) is \(B^{*j}A^{i*}\)
- The \((1, 1)\)-transposition of \[
\begin{bmatrix}
\_ & 1 \\
\_ & 1
\end{bmatrix}
\begin{bmatrix}
\_ & \_ & \_ \\
\_ & \_ & \_ \\
\_ & \_ & \_
\end{bmatrix}
\begin{bmatrix}
\_ & \_ & \_ \\
\_ & \_ & \_ \\
\_ & \_ & \_
\end{bmatrix}
\]
is \[
\begin{bmatrix}
\_ & \_ & \_ \\
\_ & \_ & \_
\end{bmatrix}
\begin{bmatrix}
\_ & \_ & \_
\end{bmatrix}
\]
- If \(c = C_1 \cdots C_n\) and \((C_k, C_{k+1})\) is an \((i, j)\)-edge pair,
\[
T^{k}_{ij}(c) = C_1 \cdots C_{k+1}C_{k}^{*j}C_{k}^{i*} \cdots C_n
\]
denotes the \((i, j)\)-transposition of \(c\) in position \(k\)
The Poset Structure of PP

\[ Z_{t,s} = \left\{ T_{i_1 r_1}^{k_1} \cdots T_{i_r j_r}^{k_r} (u) \mid u \in X_{s}^{t+1} \times Y_{t}^{s+1} \right\} \]
\[ Z_{t,s} = \left\{ T_{i_rj_r}^{k_r} \cdots T_{i_1j_1}^{k_1}(u) \mid u \in X_s^{t+1} \times Y_t^{s+1} \right\} \]

\[ \mathcal{P}_{t,s} = X_s^{t+1} \times Y_t^{s+1} \sqcup Z_{t,s} \]
\[ Z_{t,s} = \left\{ \mathcal{I}_{i_jr}^{k_r} \cdots \mathcal{I}_{i_1j_1}^{k_1} (u) \mid u \in X_{s+1}^t \times Y_{t+1}^s \right\} \]

\[ \mathcal{PP}_{t,s} = X_{s+1}^t \times Y_{t+1}^s \sqcup Z_{t,s} \]

- For \( c \in \mathcal{PP}_{t,s} \) define \( c < \mathcal{I}_{i_j}^k (c) \)
The action of $T_{i_r j_r}^{k_r} \ldots T_{i_1 j_1}^{k_1}$ on $u \in X_{s+1}^t \times Y_{t+1}^s$ uniquely determines an $(s, t)$-shuffle $\sigma$; thus we denote

$$T_\sigma (u) = T_{i_r j_r}^{k_r} \ldots T_{i_1 j_1}^{k_1} (u)$$
The action of $T_{i_rj_r}^{k_r} \cdots T_{i_1j_1}^{k_1}$ on $u \in X_{s+1}^t \times Y_{s+1}^t$ uniquely determines an $(s, t)$-shuffle $\sigma$; thus we denote

$$T_\sigma(u) = T_{i_rj_r}^{k_r} \cdots T_{i_1j_1}^{k_1}(u)$$

- Define $T_{Id} = Id$
The action of $T_{irjr}^{k_r} \cdots T_{i1j1}^{k_1}$ on $u \in X_{s+1}^{t+1} \times Y_{t+1}^{s+1}$ uniquely determines an $(s, t)$-shuffle $\sigma$; thus we denote

$$T_{\sigma}(u) = T_{irjr}^{k_r} \cdots T_{i1j1}^{k_1}(u)$$

Define $T_{Id} = Id$

$T_{\sigma}(u)$ is undefined if $\sigma$ fails to represent a composition of $(i, j)$-transpositions on $u$
The Poset Structure of PP

- The action of $\mathcal{T}_{i_rj_r} \cdots \mathcal{T}_{i_1j_1}$ on $u \in X_{s+1}^t \times Y_{t+1}^s$ uniquely determines an $(s, t)$-shuffle $\sigma$; thus we denote

\[ \mathcal{T}_\sigma(u) = \mathcal{T}_{i_rj_r} \cdots \mathcal{T}_{i_1j_1}(u) \]

- Define $\mathcal{T}_{\text{Id}} = \text{Id}$

- $\mathcal{T}_\sigma(u)$ is undefined if $\sigma$ fails to represent a composition of $(i, j)$-transpositions on $u$

- For $u_1 \leq u_2 \in X_{s+1}^t \times Y_{t+1}^s$, define $\mathcal{T}_\sigma(u_1) \leq \mathcal{T}_\sigma(u_2)$ if either
The Poset Structure of PP

- The action of $T_{i_r} T_{i_1} \cdots T_{i_r} T_{i_1}$ on $u \in X^{t+1}_s \times Y^{s+1}_t$ uniquely determines an $(s, t)$-shuffle $\sigma$; thus we denote
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The Poset Structure of PP

- The action of $T_{i_r j_r}^{k_r} \cdots T_{i_1 j_1}^{k_1}$ on $u \in X_s^{t+1} \times Y_t^{s+1}$ uniquely determines an $(s, t)$-shuffle $\sigma$; thus we denote

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  - $(u_1, u_2)$ is an edge of $X_s^{t+1} \times Y_t^{s+1}$ or
  - $u_2$ is “$\sigma$-compatible” with $u_1$
The Poset PP(1,2)

- There is no action of $T$ on $u_2 = \left[ \begin{array}{c} \top \top \\ \top \top \end{array} \right] \left[ \begin{array}{c} \top \top \\ \top \top \\top \top \end{array} \right] [\gamma \gamma \gamma]$
The Poset PP(1,2)

- There is no action of $\mathcal{T}$ on $u_2 = \left[ \begin{array}{c} \gamma \\ \gamma \\ 1 \end{array} \right] \left[ \begin{array}{c} \gamma \\ \gamma \\ \gamma \end{array} \right]$.

- The action of $\mathcal{T}$ on $u_1 = \left[ \begin{array}{c} \gamma \\ \gamma \\ 1 \end{array} \right] \left[ \begin{array}{c} \gamma \\ \gamma \\ \gamma \end{array} \right]$ and $u_3 = \left[ \begin{array}{c} \gamma \\ \gamma \\ 1 \end{array} \right] \left[ \begin{array}{c} 1 \\ \gamma \\ \gamma \end{array} \right]$ produces four elements of $Z_{1,2}$:

  $u_1 \xrightarrow{\mathcal{T}_{\sigma_1}} \left[ \begin{array}{c} \gamma \\ \gamma \\ 1 \end{array} \right] \left[ \begin{array}{c} \gamma \\ \gamma \\ \gamma \end{array} \right]$ and $u_3 \xrightarrow{\mathcal{T}_{\sigma_1}} \left[ \begin{array}{c} \gamma \\ \gamma \\ 1 \end{array} \right] \left[ \begin{array}{c} 1 \\ \gamma \\ \gamma \end{array} \right]$
The Digraph of PP(1,2)

One checks that

- only $u_1$ is $\sigma_1$-compatible with $u_1$
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- only $u_1$ is $\sigma_1$-compatible with $u_1$
- $u_2$ and $u_3$ are $\sigma_2$-compatible with $u_1$
One checks that

- only $u_1$ is $\sigma_1$-compatible with $u_1$
- $u_2$ and $u_3$ are $\sigma_2$-compatible with $u_1$
- Thus $u_1 < u_3$ implies $T_{\sigma_2}(u_1) < T_{\sigma_2}(u_3)$
Vertices in Associahedra

\[ \mathcal{V}(K_2) = [\varepsilon] \leftrightarrow [\gamma] \]
Vertices in Associahedra

- \( V(K_2) = [\nearrow] \leftrightarrow [\Uparrow] \)

- \( V(K_3) = \{[\nearrow] [\nearrow \ 1], [\nearrow] [1 \ \nearrow]\} \leftrightarrow \left\{ \left[ \begin{array}{c} \nearrow \\ 1 \end{array} \right] [\Uparrow], \left[ \begin{array}{c} 1 \\ \Uparrow \end{array} \right] [\Uparrow] \right\} \)
Vertices in Associahedra

- $\mathcal{V}(K_2) = [\ast] \leftrightarrow [\gamma]$

- $\mathcal{V}(K_3) = \{[\ast] [\ast 1], [\ast] [1 \ast]\} \leftrightarrow \begin{Bmatrix} [\gamma] \quad [\gamma], \quad [1] \quad [\gamma] \end{Bmatrix}$

- But in $\mathcal{V}(K_4)$:

$$\begin{align*}
[\ast] [1 \ast] [\ast 11] &= [\ast] [\ast 1] [1 1 \ast] \leftrightarrow \\
\begin{bmatrix} \gamma \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ \gamma \\ 1 \end{bmatrix} [\gamma] &= \begin{bmatrix} 1 \\ 1 \\ \gamma \end{bmatrix} \begin{bmatrix} \gamma \\ 1 \end{bmatrix} [\gamma] \leftrightarrow \\
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\end{align*}$$
Equivalence in the Poset $X$

For $A_1 \cdots A_s$, $A'_1 \cdots A'_s \in X^t$
For $A_1 \cdots A_s$, $A'_1 \cdots A'_s \in X^t$

Compose $i^{th}$ rows $A_{i,1} \cdots A_{i,s}$ and $A'_{i,1} \cdots A'_{i,s}$
For $A_1 \cdots A_s, A'_1 \cdots A'_s \in X_s^t$

Compose $i^{th}$ rows $A_{i,1} \cdots A_{i,s}$ and $A'_{i,1} \cdots A'_{i,s}$

Obtain up-rooted binary trees $a_i, a'_i \in \wedge_s$
Equivalence in the Poset $X$

- For $A_1 \cdots A_s$, $A'_1 \cdots A'_s \in X_s^t$

- Compose $i^{th}$ rows $A_{i,1} \cdots A_{i,s}$ and $A'_{i,1} \cdots A'_{i,s}$

- Obtain up-rooted binary trees $a_i, a'_i \in \wedge_s$

- Define $a' \sim a$ if $a_i \cong a'_i$ (as planar trees) for all $i$
Dually, for $B_t \cdots B_1$, $B'_t \cdots B'_1 \in Y_t^s$
Dually, for $B_t \cdots B_1, B'_t \cdots B'_1 \in Y_t^s$

Compose $j^{th}$ columns $B_{t,j} \cdots B_{1,j}$ and $B'_{t,j} \cdots B'_{1,j}$
Equivalence in the Poset $Y$

- Dually, for $B_t \cdots B_1$, $B'_t \cdots B'_1 \in Y^s_t$
- Compose $j^{th}$ columns $B_{t,j} \cdots B_{1,j}$ and $B'_{t,j} \cdots B'_{1,j}$
- Obtain down-rooted binary trees $b_j, b'_j \in \vee_t$
Dually, for $B_t \cdots B_1$, $B'_t \cdots B'_1 \in Y_t^s$

Compose $j^{th}$ columns $B_{t,j} \cdots B_{1,j}$ and $B'_{t,j} \cdots B'_{1,j}$

Obtain down-rooted binary trees $b_j$, $b'_j \in \vee_t$

Define $b' \sim b$ if $b'_j \cong b_j$ (as planar trees) for all $j$
Equivalence in the Poset PP

For \( a \times b, \ c \times d \in X_{s+1}^t \times Y_{s+1}^t \)
For $a \times b, c \times d \in X_{s}^{t+1} \times Y_{t}^{s+1}$

Define $a \times b \sim c \times d$ if $a \sim c$ and $b \sim d$
Equivalence in the Poset PP

- For $a \times b, c \times d \in X_s^{t+1} \times Y_t^{s+1}$

- Define $a \times b \sim c \times d$ if $a \sim c$ and $b \sim d$

- If $u_1 \sim u_2$ and either $(u_1, u_2)$ is an edge or $u_2$ is $\sigma$-compatible with $u_1$ define
  \[ \mathcal{T}_\sigma(u_1) \sim \mathcal{T}_\sigma(u_2) \]
For $a \times b, c \times d \in X_{s}^{t+1} \times Y_{t}^{s+1}$

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If $u_1 \sim u_2$ and either $(u_1, u_2)$ is an edge or $u_2$ is $\sigma$-compatible with $u_1$ define

$$\mathcal{I}_\sigma(u_1) \sim \mathcal{I}_\sigma(u_2)$$

Then $\mathcal{K}\mathcal{K}_{t+1,s+1} = \mathcal{P}\mathcal{P}_{t,s} / \sim$
For $a \times b, c \times d \in X_s^{t+1} \times Y_t^{s+1}$

- Define $a \times b \sim c \times d$ if $a \sim c$ and $b \sim d$

- If $u_1 \sim u_2$ and either $(u_1, u_2)$ is an edge or $u_2$ is $\sigma$-compatible with $u_1$

  - define

    $$\mathcal{I}_\sigma(u_1) \sim \mathcal{I}_\sigma(u_2)$$

- Then $\mathcal{KK}_{t+1,s+1} = \mathcal{PP}_{t,s}/\sim$

- $\vartheta : \mathcal{PP}_{t,s} \rightarrow \mathcal{KK}_{t+1,s+1}$ denotes the projection
1. Construct $PP_{t,s}$ as a subdivision of $P_{s+t}$ in two steps
Constructing Matrahedra

1. Construct $PP_{t,s}$ as a subdivision of $P_{s+t}$ in two steps

- **Step 1**: Replace codim 1 cell $s| (t+s) \subset P_{s+t}$ with $\Delta_P^{(t)} (\cup s_{s+1}) \times \Delta_P^{(s)} (\gamma^{t+1})$

$$P_{2+1} \supset 12|3 \leftarrow \Delta_P^{(1)} (\cup 3) \times \Delta_P^{(2)} (\gamma)$$
$P_{2+2} \supset 12|34 \leftarrow \Delta_P^{(2)}(\Lambda_3) \times \Delta_P^{(2)}(\Gamma^3)$
Constructing Matrahedra

\[ \Delta_P^3 (\Uparrow) \times \Delta_P^1 (\gamma^4) \rightarrow 1|234 \subset P_{1+3} \]
Step 2: Use $T_{\sigma}$ to propagate the subdivision of $s \mid (t + s)$ to remaining cells.
Step 2: Use $T_\sigma$ to propagate the subdivision of $s| (t + s)$ to remaining cells

$PP_{3,1}$
Constructing Matrahedra

2. \( KK_{t+1,s+1} = PP_{t,s}/\sim \)
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- Graphs associated with equivalent vertices are isomorphic (forgetting levels)
Constructing Matrahedra

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- Graphs associated with equivalent vertices are isomorphic (forgetting levels)

- If all vertices of a face $e \subset PP_{t,s}$ are equivalent, then

  $$\vartheta \vartheta (e) = \vartheta \vartheta (\mathcal{V}(e)) \in \mathcal{V}(KK_{t+1,s+1})$$
Constructing Matrahedra

2. \( KK_{t+1,s+1} = PP_{t,s} / \sim \)

- Graphs associated with equivalent vertices are isomorphic (forgetting levels)

- If all vertices of a face \( e \subset PP_{t,s} \) are equivalent, then
  \[
  \vartheta \vartheta (e) = \vartheta \vartheta (\mathcal{V}(e)) \in \mathcal{V}(KK_{t+1,s+1})
  \]

- Strings of matrices associated with the boundary components of a single cell \( e \subset KK \) determine a string of matrices associated with \( e \)
The Matrehedron KK(2,2)
The Projection from PP(3,1) to KK(4,2)
A Degenerate Square in PP(3,1)
The Matrahedron KK(4,2) as a Subdivision of the Cube
The 2-Faces of KK(4,2)
The **free matrad** $\mathcal{H}_\infty$ is the bigraded module generated by all classes of formal matrix products associated with cells of $KK$.
The Free Matrad

- The *free matrad* $\mathcal{H}_\infty$ is the bigraded module generated by all classes of formal matrix products associated with cells of $KK$

- $(\mathcal{H}_\infty)_{1,1} = \langle 1 \rangle$; $(\mathcal{H}_\infty)_{1,2} = \langle \eta \rangle$; $(\mathcal{H}_\infty)_{2,1} = \langle \gamma \rangle$
The free matrad $\mathcal{H}_\infty$ is the bigraded module generated by all classes of formal matrix products associated with cells of $KK$

$(\mathcal{H}_\infty)_{1,1} = \langle 1 \rangle; (\mathcal{H}_\infty)_{1,2} = \langle \wedge \rangle; (\mathcal{H}_\infty)_{2,1} = \langle \gamma \rangle$

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The Free Matrad

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- $\theta_t^s$ is associated with the top dim’l cell $e_{t,s}$ of $KK_{t,s}$
The free matrad $\mathcal{H}_\infty$ is the bigraded module generated by all classes of formal matrix products associated with cells of $KK$

$$(\mathcal{H}_\infty)_{1,1} = \langle 1 \rangle; \quad (\mathcal{H}_\infty)_{1,2} = \langle \forall \rangle; \quad (\mathcal{H}_\infty)_{2,1} = \langle \forall \rangle$$

$\theta_s^t \in (\mathcal{H}_\infty)_{t,s}$

$\theta_s^t$ is associated with the top dim’l cell $e_{t,s}$ of $KK_{t,s}$

The relations in $\mathcal{H}_\infty$ are encoded by the cellular boundary in $KK$
An $A_\infty$-bialgebra is a module $H$ together with a family of operations
$\{\theta^t_s \in \text{Hom} \left( H^\otimes s, H^\otimes t \right) \}_{s, t \geq 1}$ and a chain map

$$\varphi : \mathcal{H}_\infty \to \left\{ \text{Hom} \left( H^\otimes s, H^\otimes t \right) \right\}_{s, t \geq 1}$$

such that $\varphi \left( e_{t,s} \right) = \theta^t_s$
An $A_\infty$-bialgebra is a module $H$ together with a family of operations \( \{ \theta^t_s \in Hom(H^\otimes s, H^\otimes t) \}_{s,t\geq 1} \) and a chain map
\[
\varphi : \mathcal{H}_\infty \rightarrow \left\{ Hom\left( H^\otimes s, H^\otimes t \right) \right\}_{s,t\geq 1}
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such that $\varphi (e_{t,s}) = \theta^t_s$

Naturally occurring examples have been considered in joint work with A. Berciano and M. Vejdemo-Johansson
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Submodules $E \otimes \Gamma \subset H_* (\mathbb{Z}, n; \mathbb{Z}_p)$ are $A_\infty$-bialgebras with operations $\{\mu, \Delta, \Delta_p\}$
An $A_{\infty}$-bialgebra is a module $H$ together with a family of operations
\[ \{ \theta^t_s \in \text{Hom} \left( H^{\otimes s}, H^{\otimes t} \right) \}_{s,t \geq 1} \]
and a chain map
\[ \varphi : \mathcal{H}_\infty \rightarrow \left\{ \text{Hom} \left( H^{\otimes s}, H^{\otimes t} \right) \right\}_{s,t \geq 1} \]

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Naturally occurring examples have been considered in joint work with A. Berciano and M. Vejdemo-Johansson

Submodules $E \otimes \Gamma \subset H_\ast \left( \mathbb{Z}, n; \mathbb{Z}_p \right)$ are $A_{\infty}$-bialgebras with operations $\{ \mu, \Delta, \Delta_p \}$

$H^\ast \left( C_n; F \right)$ is an $A_{\infty}$-bialgebra with operations $\{ \mu, \mu_n, \Delta \}$
Thank you!