Matrads, Matrahedra and A_∞-Bialgebras

Joint work with Samson Saneblidze

Ron Umble, speaker
ron.umble@millersville.edu

Millersville University of Pennsylvania

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- An A_∞-*algebra* is a graded module A together with a family of operations $\{\mu_n \in Hom(A^{\otimes n}, A)\}_{n\geq 1}$ and a chain map

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- In this talk we construct a new family of polyhedra called *matrahedra* and use them to define the *free matrad* $\mathcal{H}_\infty$

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- $H_*(\Omega\Sigma X; F)$ is an A_∞-bialgebra with operations $\left\{\theta_2^1, \theta_1^n\right\}_{n\geq 2}$

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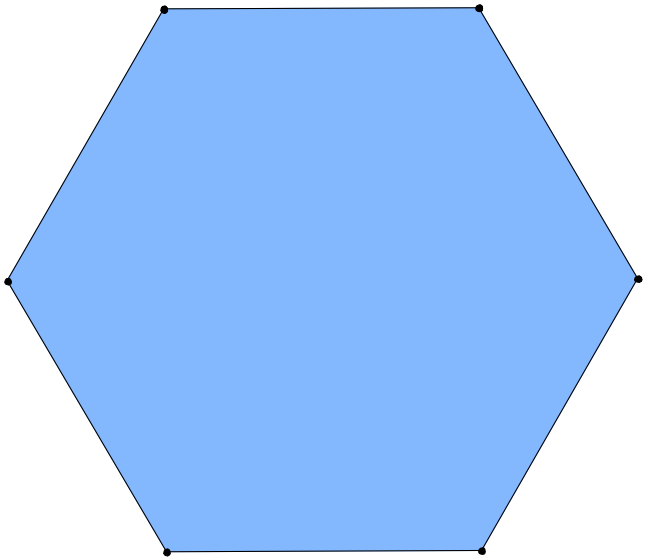

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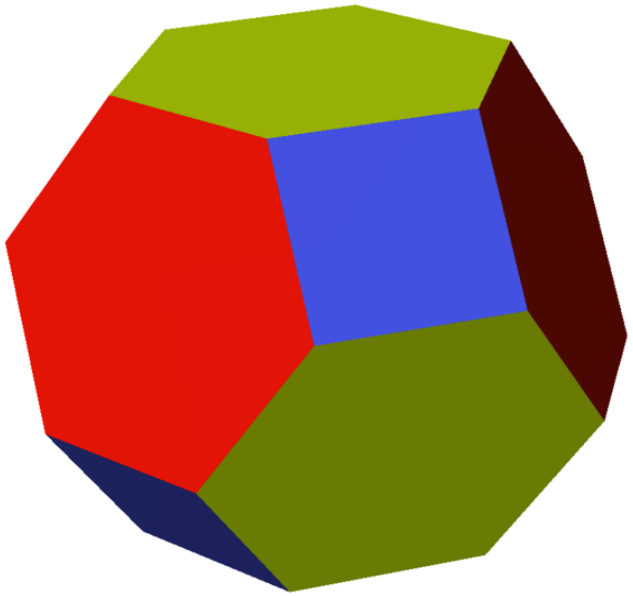

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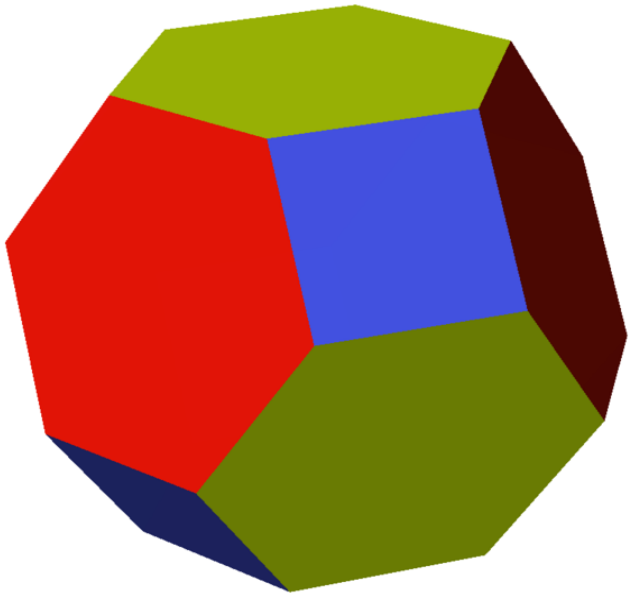

- P_n is an $(n-1)$-dim'l polyhedron

Combinatorics of Permutahedra

Let $\underline{n} = \{1, 2, \ldots n\}$

- $\{\text{Faces in codim } p\} \leftrightarrow \{\text{Partitions } U_1 | \cdots | U_{p+1} \text{ of } \underline{n}\}$

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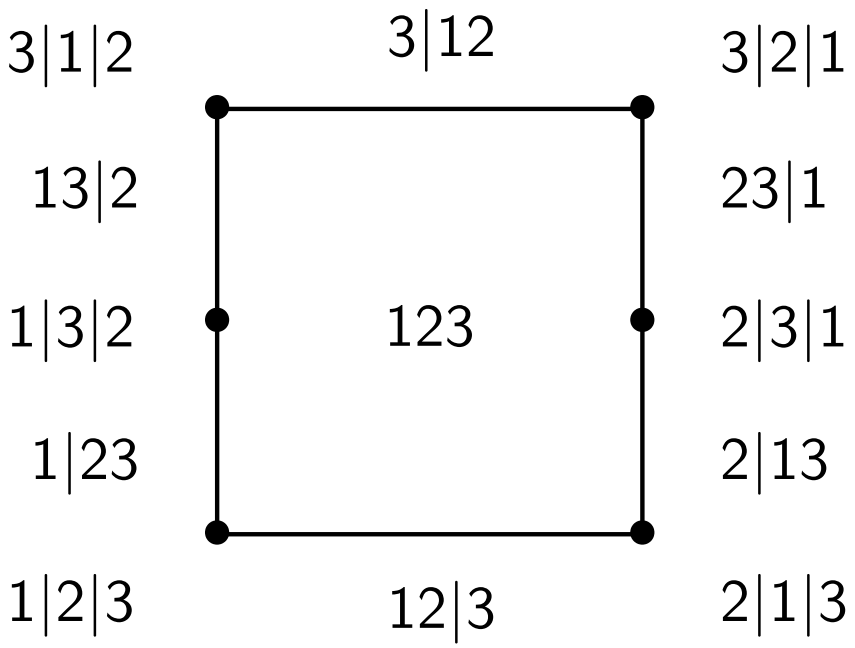
- $P_1 : \ * \leftrightarrow \{1\}$
- $P_2 : \ \text{edge} \leftrightarrow \{12\}\,; \ \text{vertices} \leftrightarrow \{1|2, 2|1\}$

Combinatorics of Permutahedra

- P_3 as a subdivision of $P_2 \times I$:



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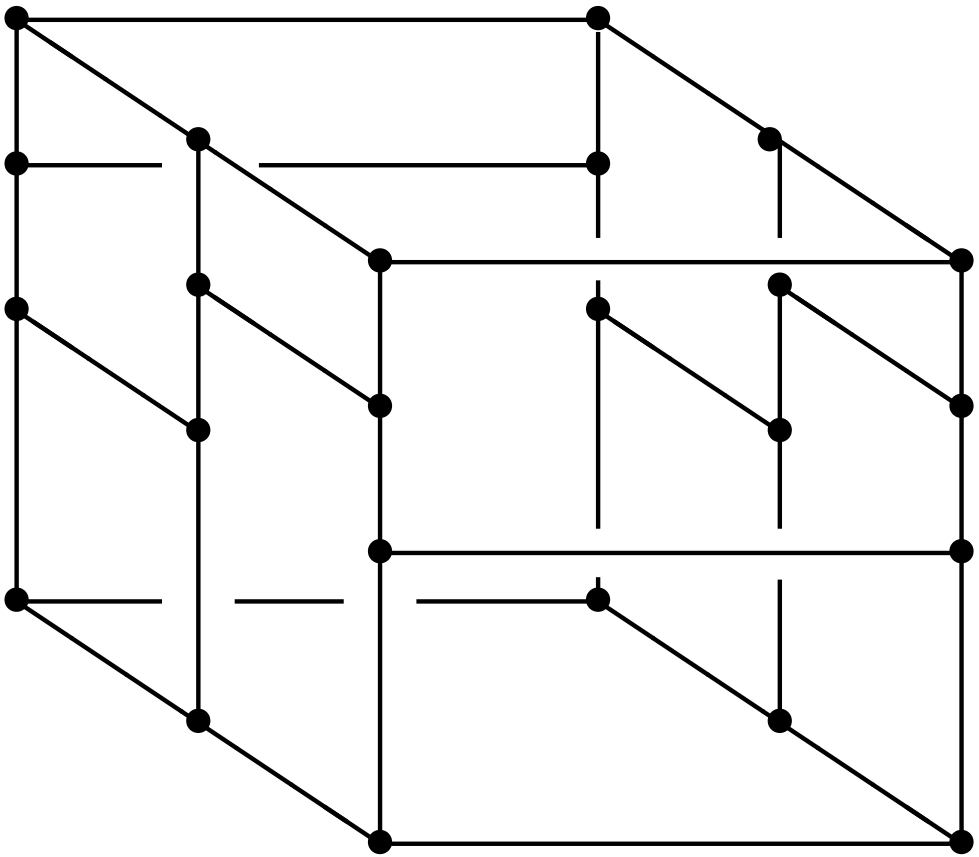

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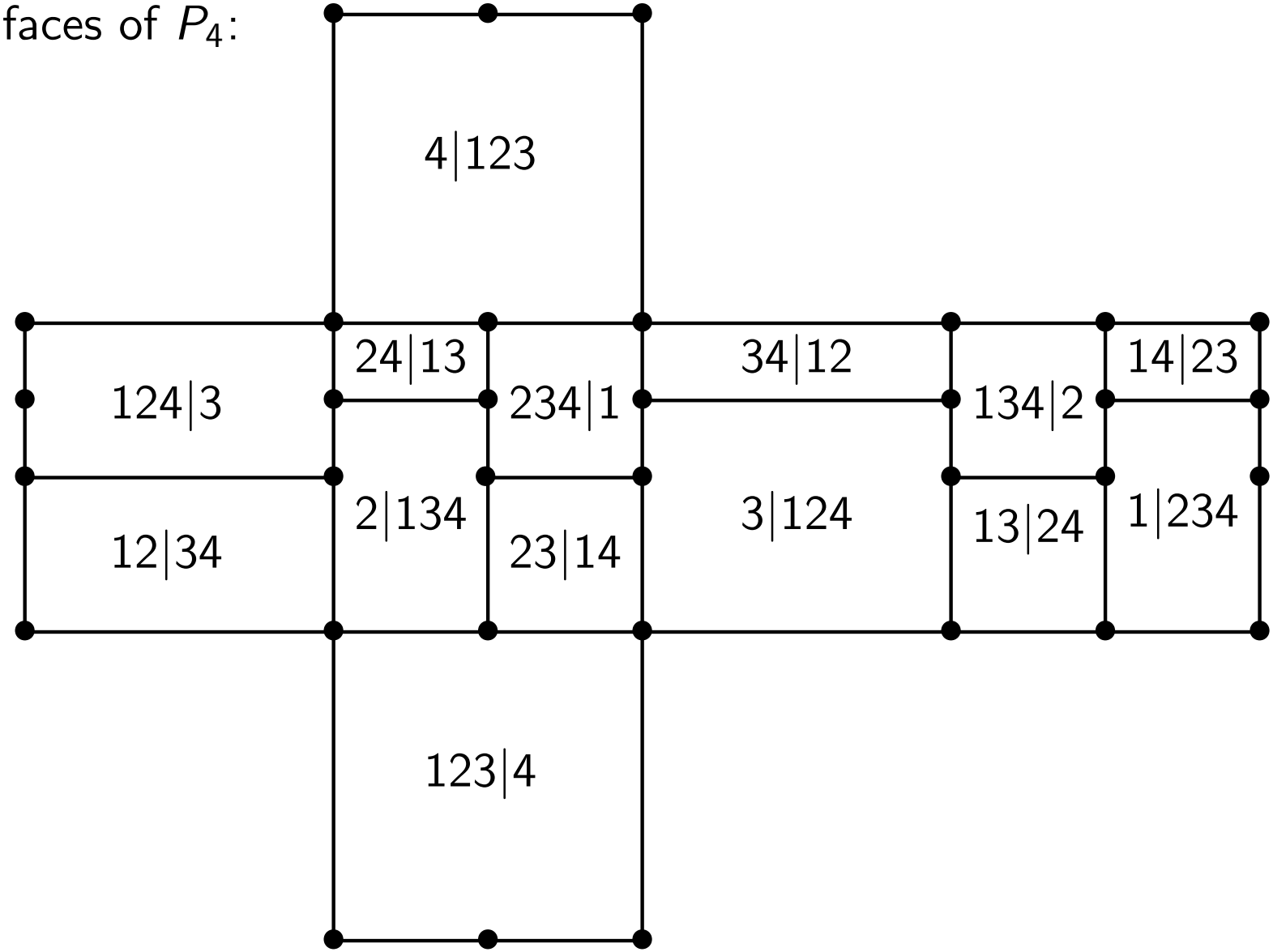
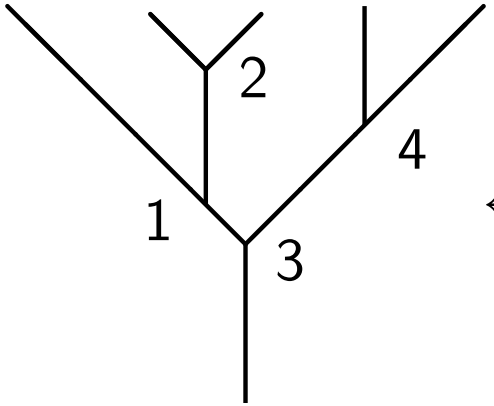
- 2-faces of P_4:



Posets of Vertices

- $\wedge_n$ and $\vee_n$ denote the sets of up-rooted and down-rooted planar *binary* trees with n levels and $n+1$ leaves



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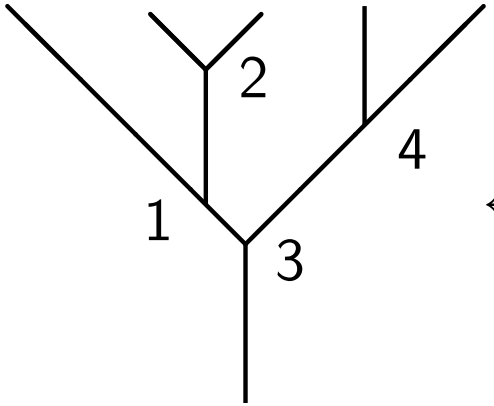

$$\longleftrightarrow \qquad 2|4|1|3 \in \vee_4$$

- The bijections $S_n \leftrightarrow \wedge_n$ and $S_n \leftrightarrow \vee_n$ transfer the *Bruhat* partial order generated by transpositions

$$a_1|\cdots|a_n < a_1|\cdots|a_{i+1}|a_i|\cdots|a_n \text{ iff } a_i < a_{i+1}$$

to $\wedge_n$ and $\vee_n$

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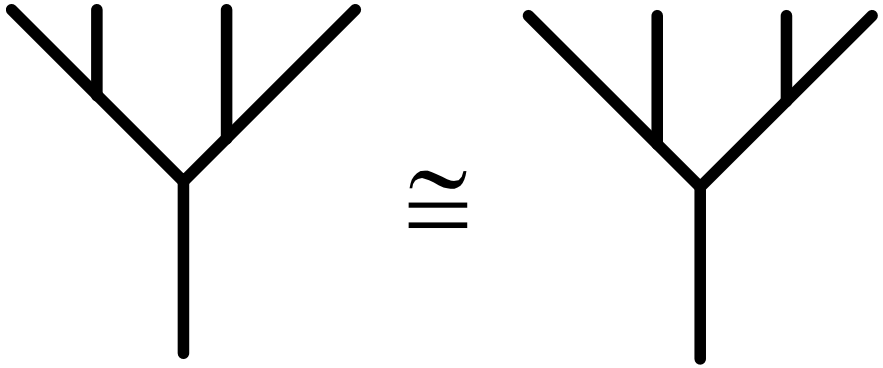
- $\vartheta(a) = \vartheta(b)$ iff $a \cong b$ (as planar trees)



1|3|2 $\cong$ 3|1|2

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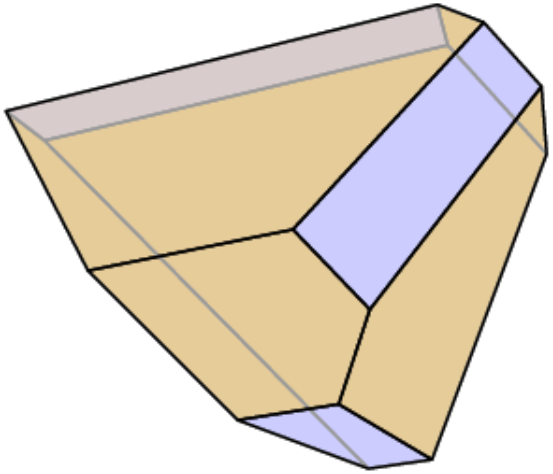

J-L Loday's rendering of K_5

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1. Construct the poset of vertices $\mathcal{PP}_{t,s}$
2. Construct $PP_{t,s}$ as a subdivision of P_{s+t}
3. Form the quotient space $KK_{t+1,s+1} = PP_{t,s}/\sim$

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- For us, all choices were made once and for all when constructing the S-U diagonal

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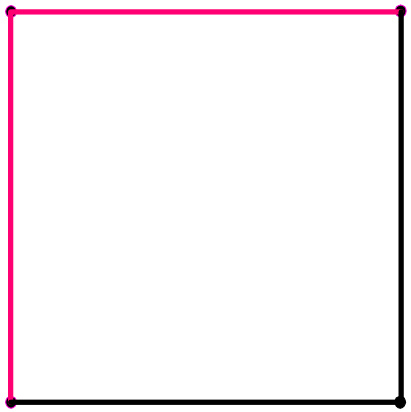
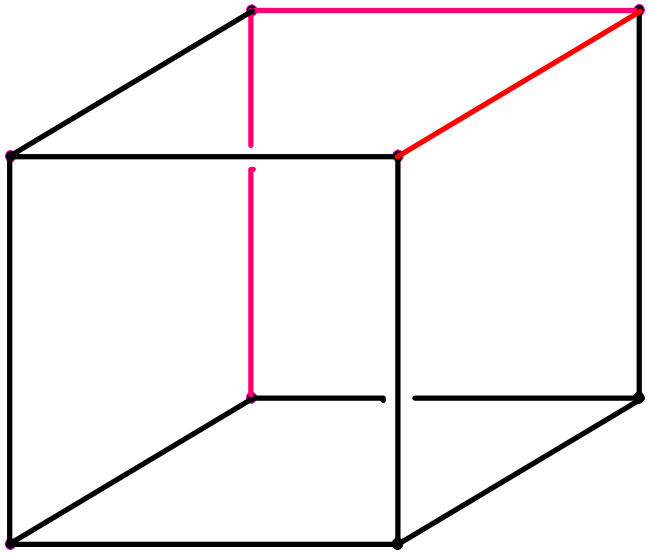
- Think of $\Delta_P^{(s-1)}(\curlyvee^{t+1})$ as a $(t-1)$-dim'l subcomplex of $P_t^{\times s}$
- $\Delta_P^{(1)}(\curlywedge) = \curlywedge \times \curlywedge = P_1^{\times 2}$ and $\Delta_P^{(1)}(\curlyvee) = \curlyvee \times \curlyvee = P_1^{\times 2}$

  (the index 2 is suppressed)

Iterated S-U Diagonal



$\Delta_P^{(1)}(\curlywedge_3) \subset P_2^{\times 2}$



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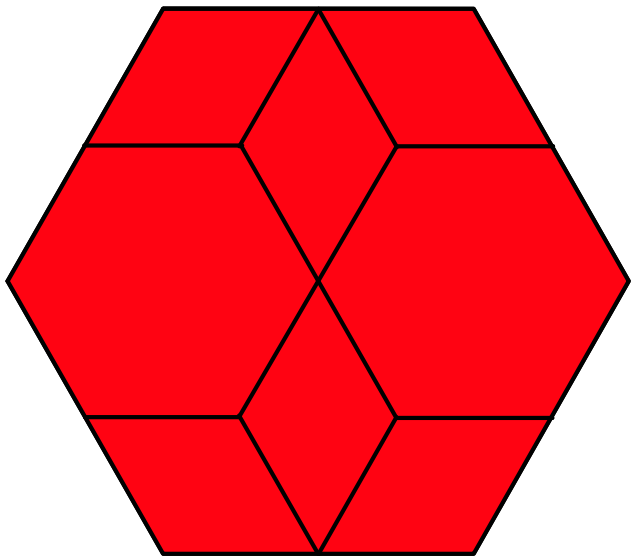

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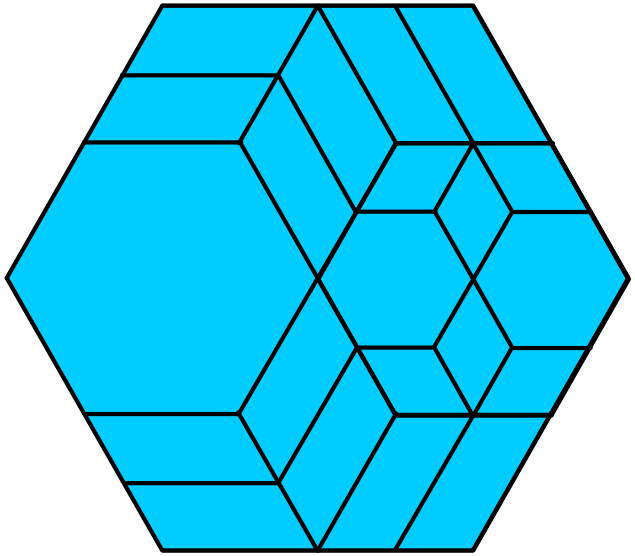

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- $X_s^t \times Y_t^s$ is a subposet of $\wedge_s^{\times t} \times \vee_t^{\times s}$

Edges of a Poset

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- $\mathcal{PP}_{t,s} = X_s^{t+1} \times Y_t^{s+1} \sqcup Z_{t,s}$

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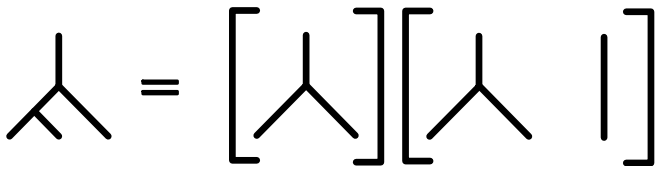

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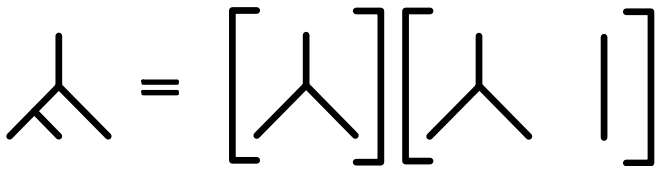

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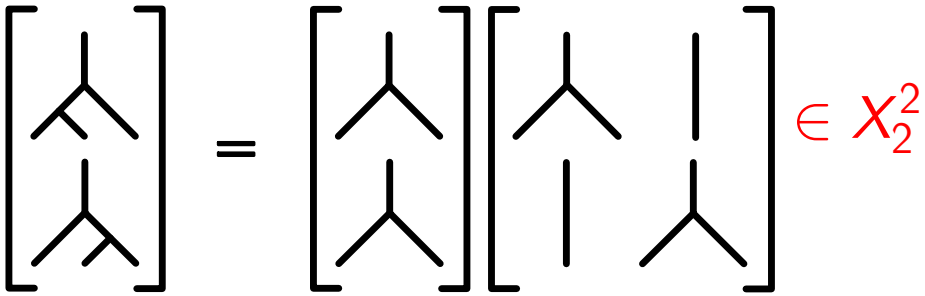

$\in X_2^2$

Vertices as Matrix Products

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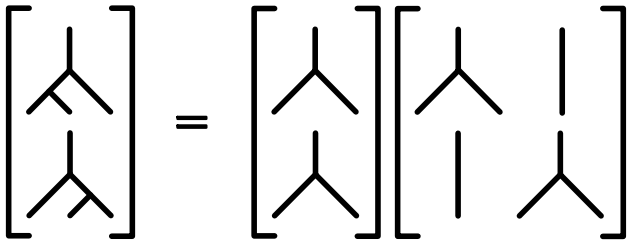

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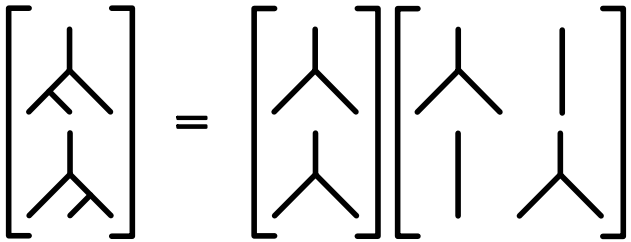

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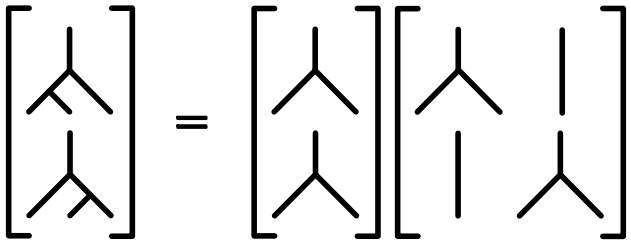

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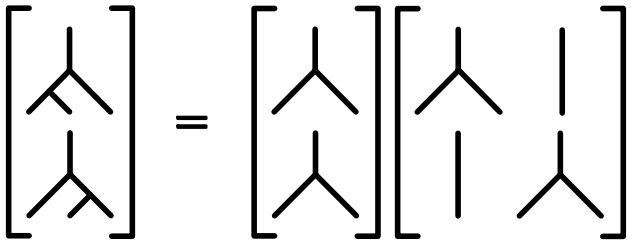

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$$\left. < u_3 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \mathbf{1} & \curlywedge \\ \mathbf{1} & \curlywedge \end{bmatrix} [\curlyvee\ \curlyvee\ \curlyvee] \right\} \subset \wedge_2^{\times 2} \times \vee_1^{\times 3}$$

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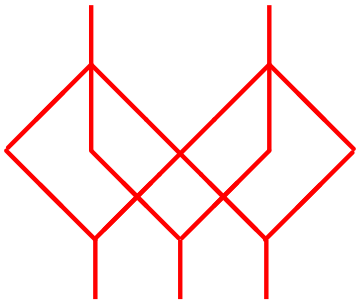
- Pictured as a non-planar upward-directed graph



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$$\overline{\mathbf{M}}_X^Y = \begin{array}{ccccc} & (M_{y_{11},x_{11}} & \otimes\cdots\otimes & M_{y_{1,p},x_{1p}}) & \\ \otimes & (M_{y_{21},x_{21}} & \otimes\cdots\otimes & M_{y_{2p},x_{2p}}) & \\ & \vdots & & \vdots & \\ \otimes & (M_{y_{q1},x_{q1}} & \otimes\cdots\otimes & M_{y_{qp},x_{qp}}) & \subset \overline{\mathbf{M}} \end{array}$$

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- Each of the q rows is a submodule of $M^{\otimes p}$

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- B is represented by the $q \times p$ matrix

$$[B] = \begin{bmatrix} g_{x_{1,1}}^{y_{1,1}} & \cdots & g_{x_{1,p}}^{y_{1,p}} \\ \vdots & & \vdots \\ g_{x_{q,1}}^{y_{q,1}} & \cdots & g_{x_{q,p}}^{y_{q,p}} \end{bmatrix}$$

with entries in G and rows identified with elements of $G^{\otimes p}$

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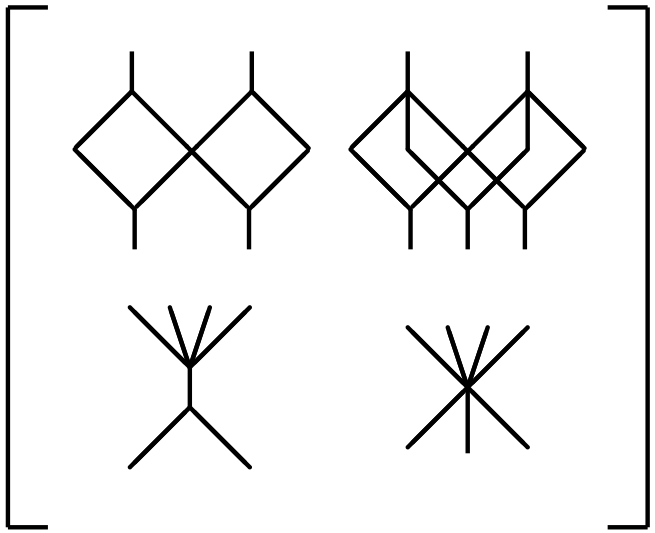
- A monomial $A \in \mathbf{M}$ is represented by a *bisequence matrix*

$$A = \begin{bmatrix} \alpha_{x_1}^{y_1} & \cdots & \alpha_{x_p}^{y_1} \\ \vdots & & \vdots \\ \alpha_{x_1}^{y_q} & \cdots & \alpha_{x_p}^{y_q} \end{bmatrix}$$

Bisequence Matrices

- We refer to A as a $q \times p$ *monomial*

$$A \in \mathbf{M}_{23}^{\substack{2\\4}} \quad \leftrightarrow$$



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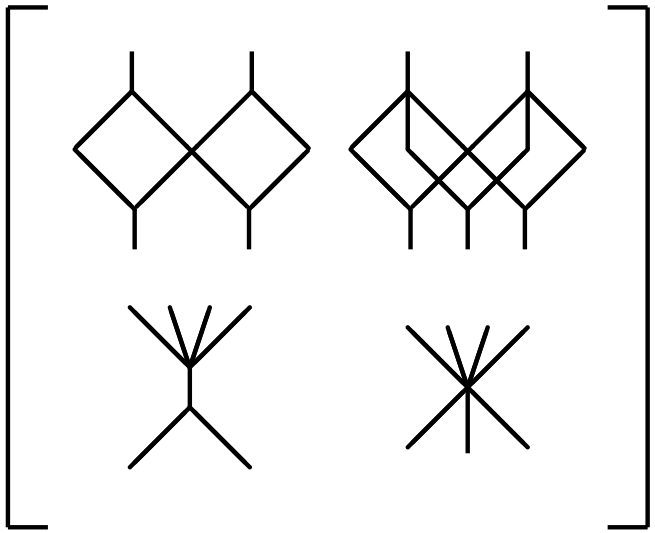

- A *Transverse Pair* (TP) of bisequence matrices has form

$$\begin{bmatrix} \alpha_p^{y_1} \\ \vdots \\ \alpha_p^{y_q} \end{bmatrix} \otimes \begin{bmatrix} \beta_{x_1}^q & \cdots & \beta_{x_p}^q \end{bmatrix} \in \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q$$

- $A \otimes B \in \overline{\mathbf{M}} \otimes \overline{\mathbf{M}}$ is a *Block Transverse Pair* (BTP) if there exist block decompositions $A = [A_{ij}]$ and $B = [B_{ij}]$ such that $A_{ij} \otimes B_{ij}$ is a TP for each (i, j)

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- An element $A \otimes B \in \mathbf{M}^{\substack{1\\5\\4\\3}}_{21} \otimes \mathbf{M}^{\substack{3\\1}}_{123}$ is a BTP via

$$A = \begin{bmatrix} \alpha^1_2 & \alpha^1_1 \\ \alpha^5_2 & \alpha^5_1 \\ \alpha^4_2 & \alpha^4_1 \\ \alpha^3_2 & \alpha^3_1 \end{bmatrix} \qquad B = \begin{bmatrix} \beta^3_1 \;\; \beta^3_2 & \beta^3_3 \\ \beta^1_1 \;\; \beta^1_2 & \beta^1_3 \end{bmatrix}$$

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- BTP decomposition is unique when it exists

Upsilon Products

- Given a map on TPs

$$\gamma = \left\{ \gamma_{\mathbf{x}}^{\mathbf{y}} : \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \to \mathbf{M}_{|\mathbf{x}|}^{|\mathbf{y}|} \right\},$$

extend to a global product $\Upsilon : \overline{\mathbf{M}} \otimes \overline{\mathbf{M}} \to \overline{\mathbf{M}}$ by setting $\Upsilon(A \otimes B) = 0$ unless $A \otimes B$ is a BTP, in which case define

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- There is unique Υ-factorization in $\overline{\mathbf{M}}$

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$$\Upsilon : \mathbf{M}^{\substack{1\\5\\4\\3}}_{21} \otimes \mathbf{M}^{\substack{3\\1}}_{123} \to \mathbf{M}^{\substack{10\\3}}_{33}$$

$$\begin{bmatrix} \alpha^1_2 & \alpha^1_1 \\ \alpha^5_2 & \alpha^5_1 \\ \alpha^4_2 & \alpha^4_1 \\ \alpha^3_2 & \alpha^3_1 \end{bmatrix} \begin{bmatrix} \beta^3_1 & \beta^3_2 & \beta^3_3 \\ \beta^1_1 & \beta^1_2 & \beta^1_3 \end{bmatrix} = \begin{bmatrix} \alpha^1_2 & & \alpha^1_1 & \\ \alpha^5_2 & \beta^3_1 \;\; \beta^3_2 & \alpha^5_1 & \beta^3_3 \\ \alpha^4_2 & & \alpha^4_1 & \\ \alpha^3_2 & \beta^1_1 \;\; \beta^1_2 & \alpha^3_1 & \beta^1_3 \end{bmatrix}$$

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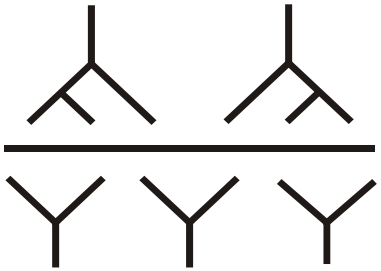
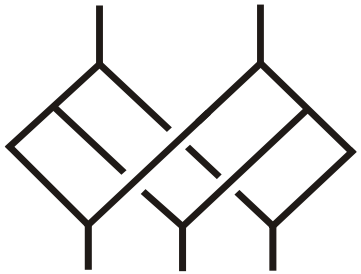
- $\begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \mathbf{1} & \curlywedge \end{bmatrix} [\Upsilon\ \Upsilon\ \Upsilon] =$  $=$ 

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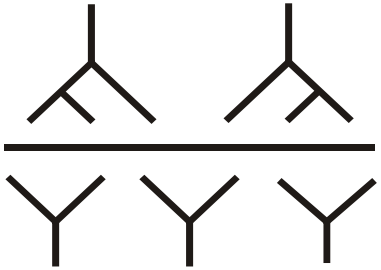
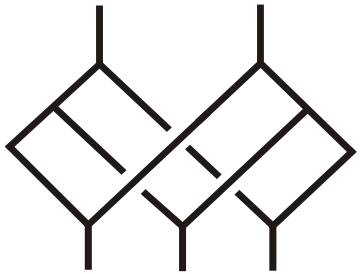
- $\begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \mathbf{1} & \curlywedge \end{bmatrix} [\Upsilon\ \Upsilon\ \Upsilon] =$  $=$ 

- The picture in the center is M. Markl's *fraction product*

Edge Pairs

- A pair or matrices (A, B) is an (i, j)-*edge pair* if

  - $A = [a_{ij}]$ is a matrix over $\{\mathbf{1}, \curlywedge\}$ whose rows contain $\curlywedge$ exactly once

  - $B = [b_{ij}]$ is a matrix over $\{\mathbf{1}, \curlyvee\}$ whose col's contain $\curlyvee$ exactly once

  - $\begin{bmatrix} a_{ij} \\ a_{i+1,j} \end{bmatrix} \begin{bmatrix} b_{ij} & b_{i,j+1} \end{bmatrix} = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlyvee & \curlyvee \end{bmatrix}$

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- (A_s, B_t) is the only possible edge pair of adjacent matrices in

$$A_1 \cdots A_s B_t \cdots B_1 \in X_s^{t+1} \times Y_t^{s+1}$$

Edge Pairs

- (x_2, y_2) is a $(1,1)$-edge pair in

$$x_1 x_2 y_2 y_1 = \begin{bmatrix} \curlywedge \\ \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} \begin{bmatrix} \curlyvee & \curlyvee & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \curlyvee \end{bmatrix} [\curlyvee \ \curlyvee \ \curlyvee]$$

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- There are no adjacent edge pairs in

$$\begin{bmatrix} \curlywedge \\ \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} \begin{bmatrix} \curlyvee & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \curlyvee & \curlyvee \end{bmatrix} [\curlyvee \ \curlyvee \ \curlyvee]$$

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- If $c = C_1 \cdots C_n$ and (C_k, C_{k+1}) is an (i, j)-edge pair,

$$\mathcal{T}_{ij}^{k}(c) = C_1 \cdots C_{k+1}^{*j} C_k^{i*} \cdots C_n$$

denotes the (i, j)-*transposition of c in position k*

The Poset Structure of PP

- $Z_{t,s} = \left\{ \mathcal{T}_{i_r j_r}^{k_r} \cdots \mathcal{T}_{i_1 j_1}^{k_1} (u) \;\middle|\; u \in X_s^{t+1} \times Y_t^{s+1} \right\}$

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- $\mathcal{PP}_{t,s} = X_s^{t+1} \times Y_t^{s+1} \sqcup Z_{t,s}$
- For $c \in \mathcal{PP}_{t,s}$ define $c < \mathcal{T}_{ij}^{k} (c)$

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- The action of $\mathcal{T}_{i_r j_r}^{k_r} \cdots \mathcal{T}_{i_1 j_1}^{k_1}$ on $u \in X_s^{t+1} \times Y_t^{s+1}$ uniquely determines an (s, t)-shuffle σ; thus we denote

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The Poset PP(1,2)

- There is no action of $\mathcal{T}$ on $u_2 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \mathbf{1} & \curlywedge \end{bmatrix} [\Upsilon \ \Upsilon \ \Upsilon]$

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- The action of $\mathcal{T}$ on

$$u_1 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} [\curlyvee\ \curlyvee\ \curlyvee] \text{ and } u_3 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \mathbf{1} & \curlywedge \\ \mathbf{1} & \curlywedge \end{bmatrix} [\curlyvee\ \curlyvee\ \curlyvee]$$

produces four elements of $Z_{1,2}$:

$$u_1 \xmapsto{\mathcal{T}_{\sigma_1}} \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} [\curlyvee\curlyvee]\,[\curlywedge\ \mathbf{1}] \xmapsto{\mathcal{T}_{\sigma_2}} [\curlyvee][\curlywedge]\,[\curlywedge\ \mathbf{1}]$$

$$u_3 \xmapsto{\mathcal{T}_{\sigma_1}} \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} [\curlyvee\curlyvee]\,[\mathbf{1}\ \curlywedge] \xmapsto{\mathcal{T}_{\sigma_2}} [\curlyvee][\curlywedge]\,[\mathbf{1}\ \curlywedge]$$

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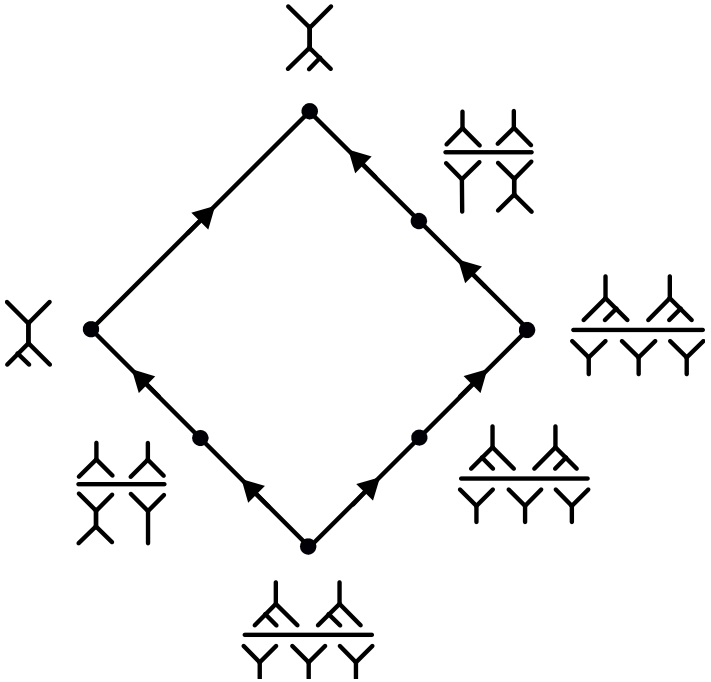
- Thus $u_1 < u_3$ implies $\mathcal{T}_{\sigma_2}(u_1) < \mathcal{T}_{\sigma_2}(u_3)$



Vertices in Associahedra

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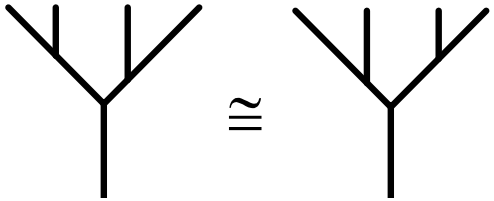
- But in $\mathcal{V}(K_4)$:

$$[\curlywedge][\mathbf{1}\ \curlywedge][\curlywedge\ \mathbf{1}\ \mathbf{1}] = [\curlywedge][\curlywedge\ \mathbf{1}][\mathbf{1}\ \mathbf{1}\ \curlywedge] \leftrightarrow$$

$$\begin{bmatrix} \curlyvee \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \curlyvee \end{bmatrix} [\curlyvee] = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \curlyvee \end{bmatrix} \begin{bmatrix} \curlyvee \\ \mathbf{1} \end{bmatrix} [\curlyvee] \leftrightarrow$$



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- $\vartheta\vartheta : \mathcal{PP}_{t,s} \to \mathcal{KK}_{t+1,s+1}$ denotes the projection

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1. Construct $PP_{t,s}$ as a subdivision of P_{s+t} in two steps

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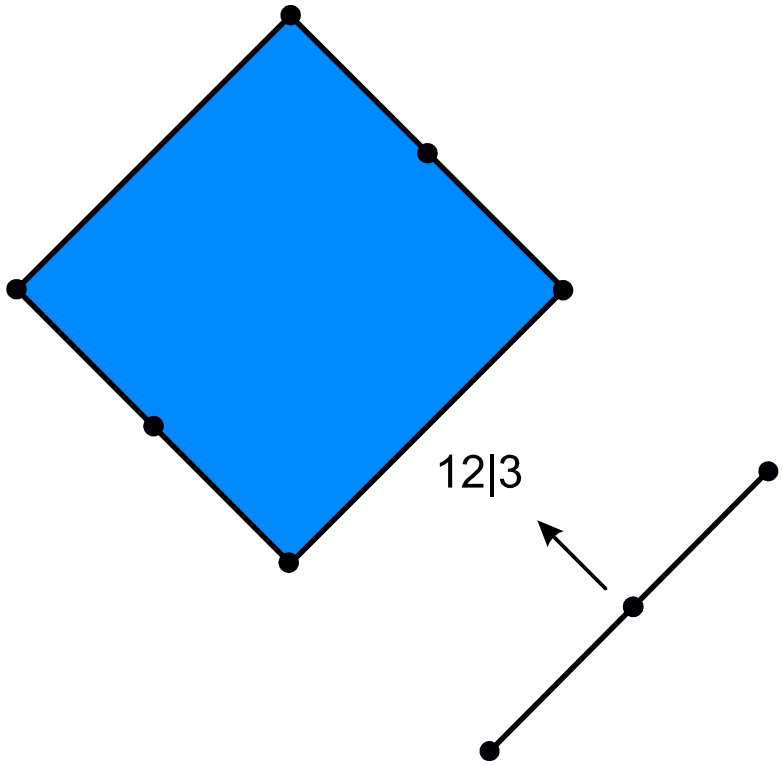
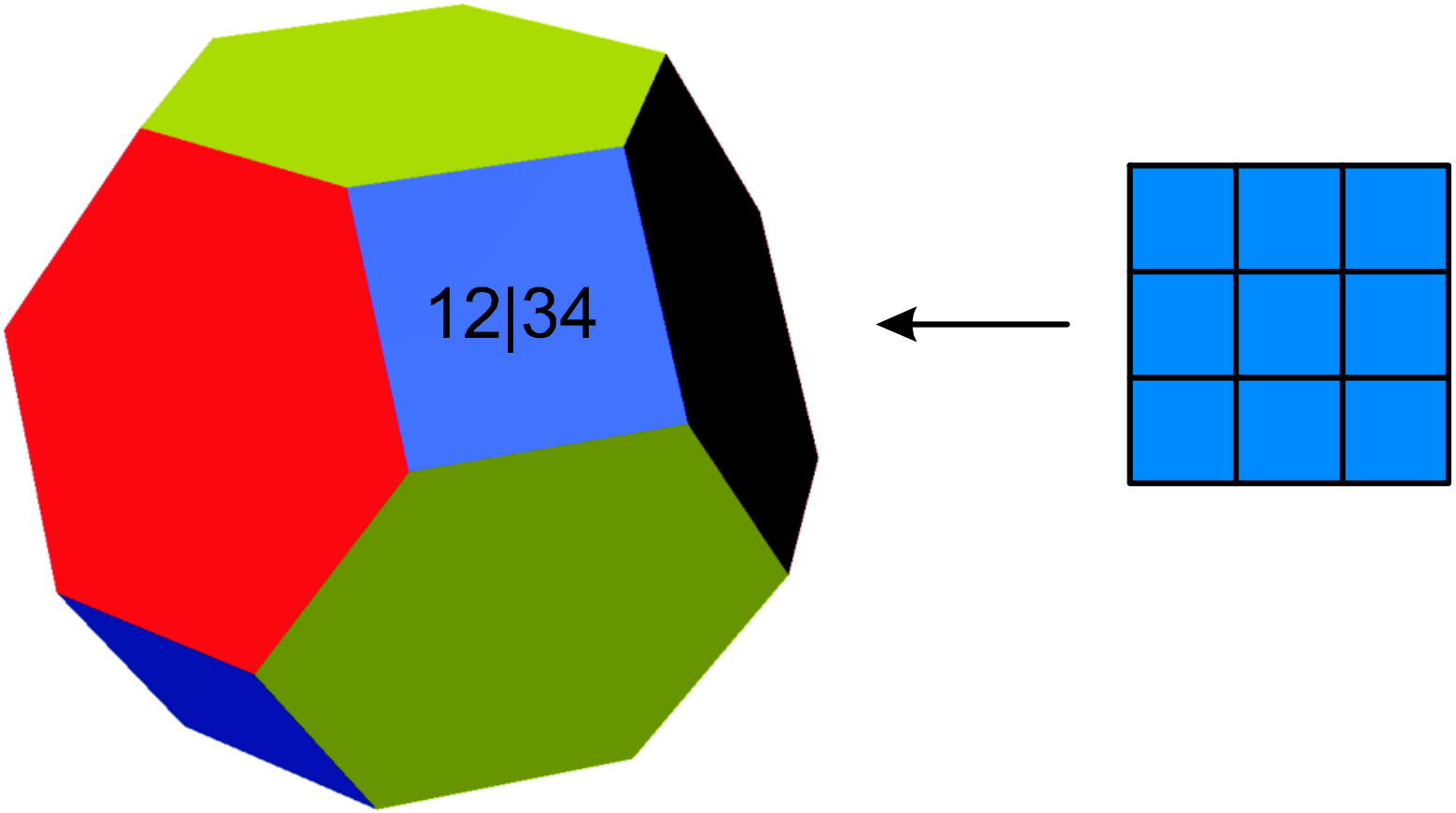
   - Step 1: Replace codim 1 cell $\underline{s}|\,(\underline{t}+s) \subset P_{s+t}$ with $\Delta_P^{(t)}(\curlywedge_{s+1}) \times \Delta_P^{(s)}(\curlyvee^{t+1})$



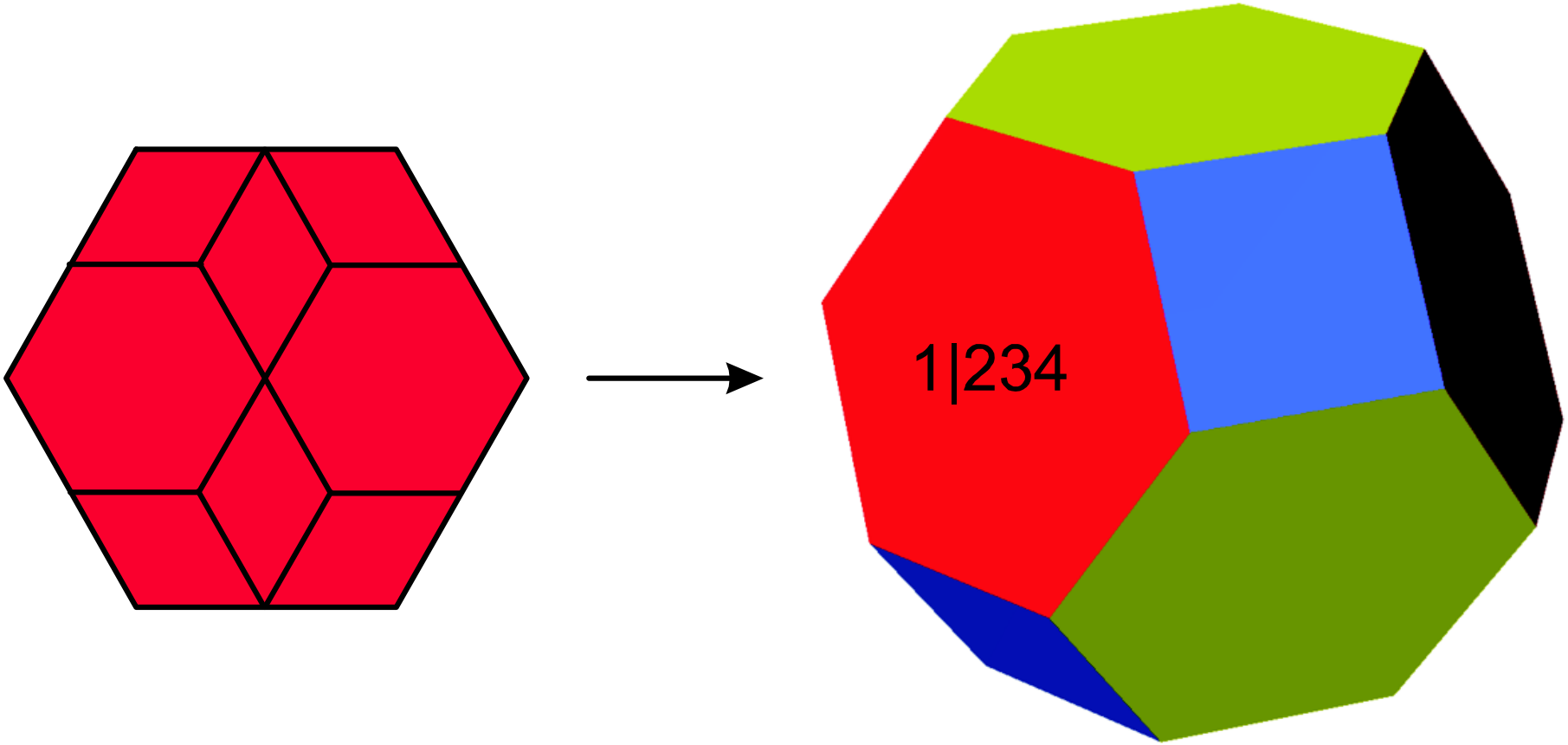
$$P_{2+1} \supset 12|3 \leftarrow \Delta_P^{(1)}(\curlywedge_3) \times \Delta_P^{(2)}(\curlyvee)$$

Constructing Matrahedra



$$P_{2+2} \supset 12|34 \leftarrow \Delta_P^{(2)}\left(\curlywedge_3\right) \times \Delta_P^{(2)}\left(\Upsilon^3\right)$$

Constructing Matrahedra



$$\Delta_P^{(3)}(\curlywedge) \times \Delta_P^{(1)}(\Upsilon^4) \to 1|234 \subset P_{1+3}$$

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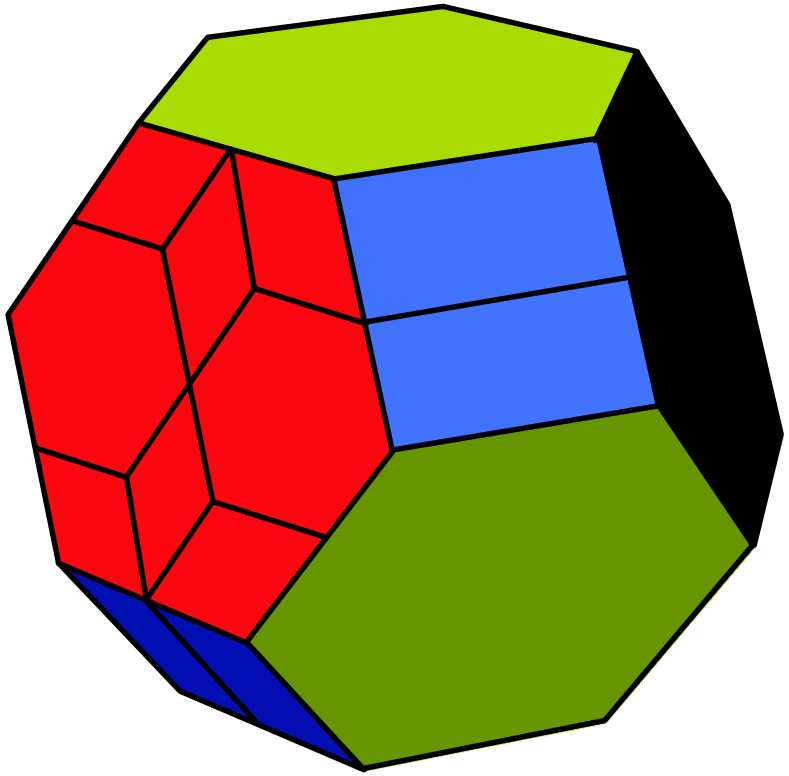

$PP_{3,1}$

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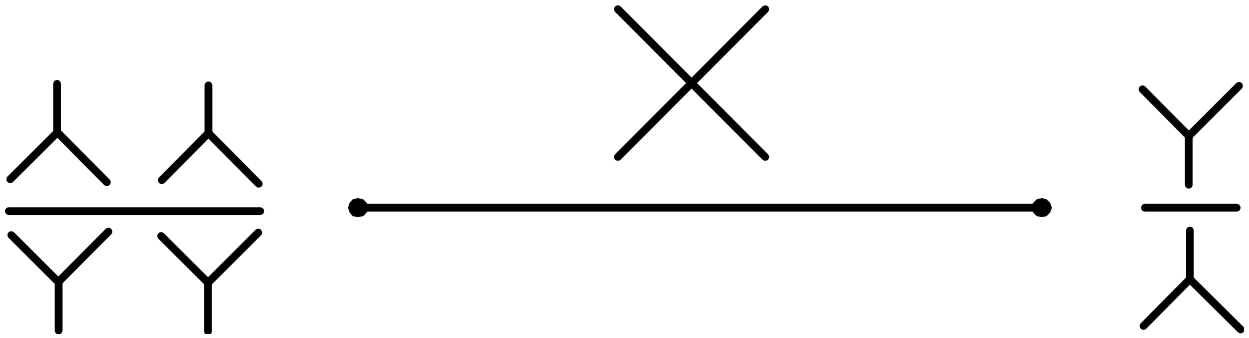
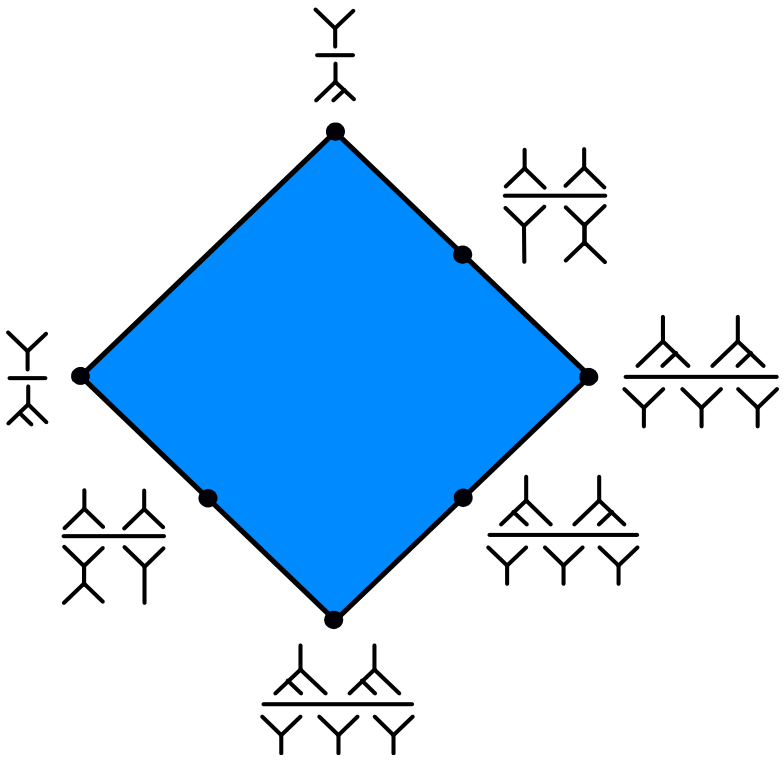
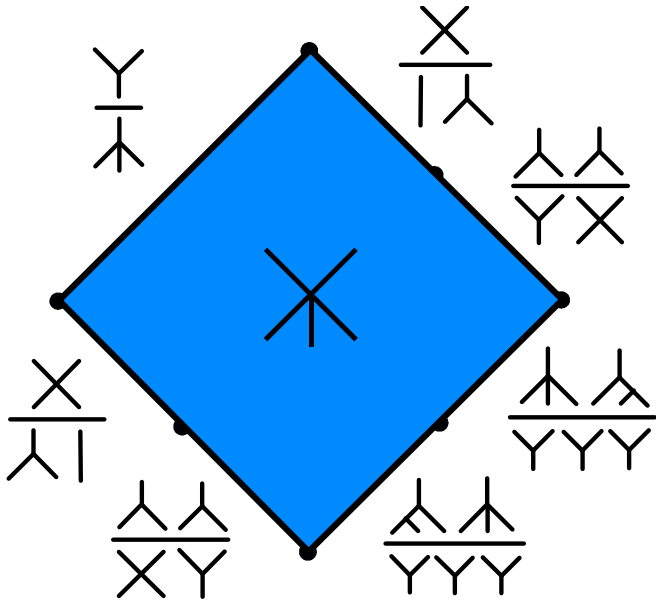
- Strings of matrices associated with the boundary components of a single cell $e \subset KK$ determine a string of matrices associated with e

The Matrehedron KK(2,2)

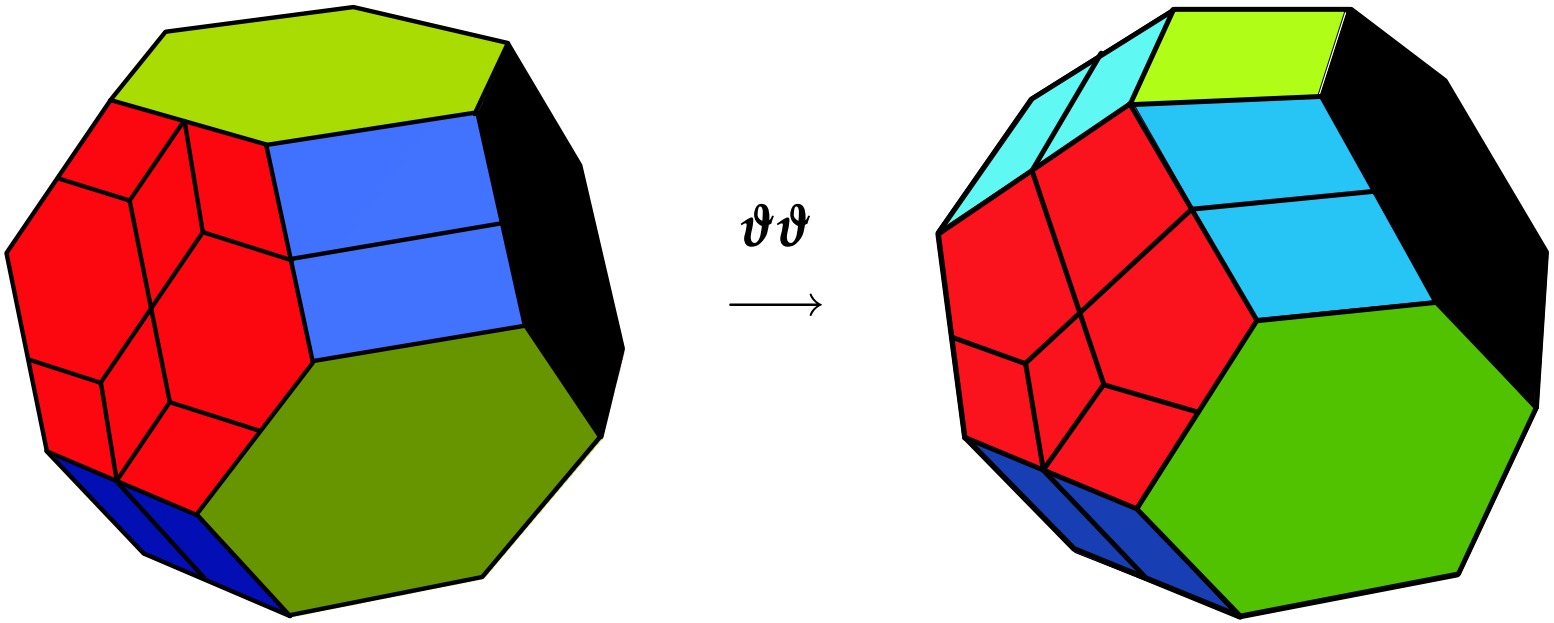


The Matrehedron KK(2,3)

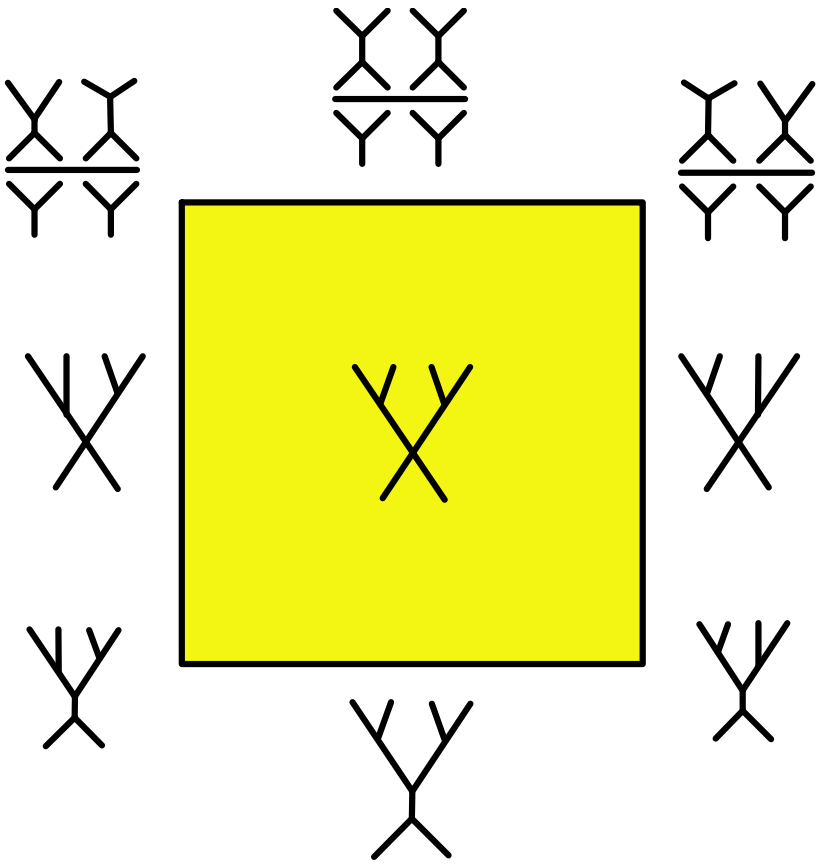




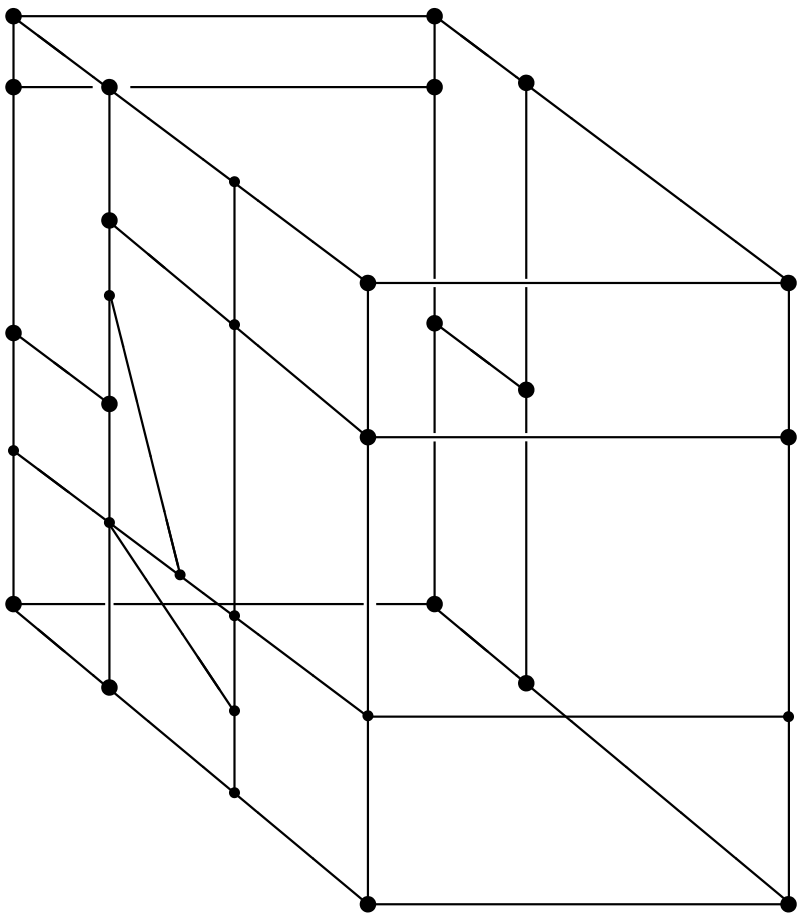
The Projection from PP(3,1) to KK(4,2)



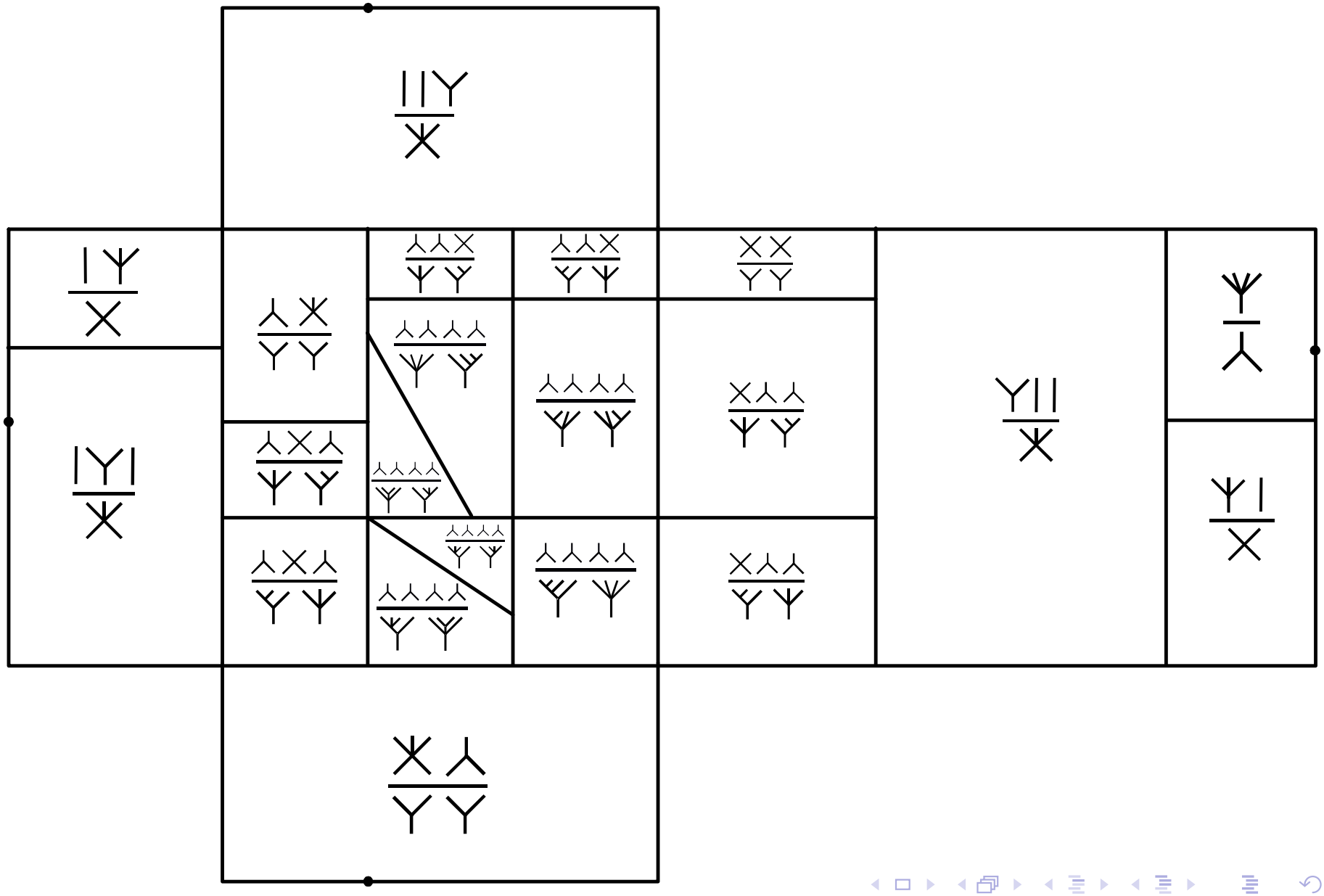
A Degenerate Square in PP(3,1)



The Matrahedron KK(4,2) as a Subdivision of the Cube



The 2-Faces of KK(4,2)



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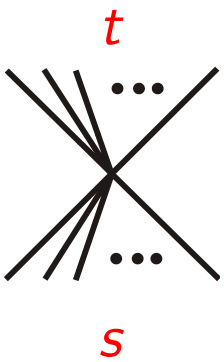
- $\theta^t_s =$ 

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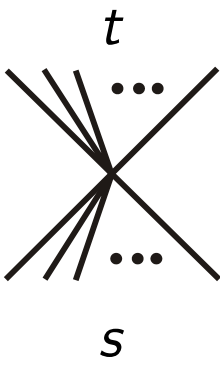
- $\theta_s^t =$ 

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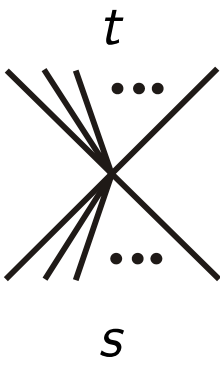
- $\theta_s^t =$ 

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- The relations in $\mathcal{H}_\infty$ are encoded by the cellular boundary in KK

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- An A_∞-*bialgebra* is a module H together with a family of operations $\left\{\theta_s^t \in Hom\left(H^{\otimes s}, H^{\otimes t}\right)\right\}_{s,t\geq 1}$ and a chain map

$$\varphi : \mathcal{H}_\infty \to \left\{Hom\left(H^{\otimes s}, H^{\otimes t}\right)\right\}_{s,t\geq 1}$$

such that $\varphi\left(e_{t,s}\right) = \theta_s^t$

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The End

Thank you!