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Cyclic Groups

Cyclic groups are groups in which every element is a power of some fixed element. (If the group is abelian and I'm using + as the operation, then I should say instead that every element is a *multiple* of some fixed element.) Here are the relevant definitions.

Definition. Let G be a group, $g \in G$. The **order** of g is the smallest positive integer n such that $g^n = 1$. If there is no positive integer n such that $g^n = 1$, then g has **infinite order**.

In the case of an abelian group with + as the operation and 0 as the identity, the order of g is the smallest positive integer n such that $ng = 0$.

Definition. If G is a group and $g \in G$, then the **subgroup generated by** g is

$$\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}.$$

If the group is abelian and I'm using + as the operation, then

$$\langle g \rangle = \{ng \mid n \in \mathbb{Z}\}.$$

Definition. A group G is **cyclic** if $G = \langle g \rangle$ for some $g \in G$. g is a **generator** of $\langle g \rangle$.

If a generator g has order n, $G = \langle g \rangle$ is **cyclic of order** n. If a generator g has infinite order, $G = \langle g \rangle$ is **infinite cyclic**.

Example. (**The integers and the integers mod n are cyclic**) Show that $\mathbb{Z}$ and $\mathbb{Z}_n$ for $n > 0$ are cyclic.

$\mathbb{Z}$ is an infinite cyclic group, because every element is a multiple of 1 (or of -1). For instance, $117 = 117 \cdot 1$.

(Remember that "$117 \cdot 1$" is really shorthand for $1 + 1 + \cdots + 1$ — 1 added to itself 117 times.)

In fact, it is the only infinite cyclic group up to **isomorphism**.

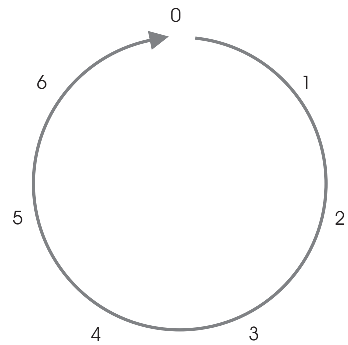
Notice that a cyclic group can have more than one generator.

If n is a positive integer, $\mathbb{Z}_n$ is a cyclic group of order n generated by 1.

For example, 1 generates $\mathbb{Z}_7$, since

$$\begin{aligned} 1 + 1 &= 2 \\ 1 + 1 + 1 &= 3 \\ 1 + 1 + 1 + 1 &= 4 \\ 1 + 1 + 1 + 1 + 1 &= 5 \\ 1 + 1 + 1 + 1 + 1 + 1 &= 6 \\ 1 + 1 + 1 + 1 + 1 + 1 + 1 &= 0 \end{aligned}$$

In other words, if you add 1 to itself repeatedly, you eventually cycle back to 0.



a cyclic group of order 7

Notice that 3 also generates $\mathbb{Z}_7$:

$$\begin{aligned} 3+3 &= 6 \\ 3+3+3 &= 2 \\ 3+3+3+3 &= 5 \\ 3+3+3+3+3 &= 1 \\ 3+3+3+3+3+3 &= 4 \\ 3+3+3+3+3+3+3 &= 0 \end{aligned}$$

The "same" group can be written using multiplicative notation this way:

$$\mathbb{Z}_7 = \{1, a, a^2, a^3, a^4, a^5, a^6\}.$$

In this form, a is a generator of $\mathbb{Z}_7$.
It turns out that in $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$, every nonzero element generates the group.
On the other hand, in $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, only 1 and 5 generate. □

Lemma. Let $G = \langle g \rangle$ be a finite cyclic group, where g has order n. Then the powers $\{1, g, \ldots, g^{n-1}\}$ are distinct.

Proof. Since g has order n, g, g^2, $\ldots g^{n-1}$ are all different from 1.

Now I'll show that the powers $\{1, g, \ldots, g^{n-1}\}$ are distinct. Suppose $g^i = g^j$ where $0 \le j < i < n$. Then $0 < i - j < n$ and $g^{i-j} = 1$, contrary to the preceding observation.

Therefore, the powers $\{1, g, \ldots, g^{n-1}\}$ are distinct. □

Lemma. Let $G = \langle g \rangle$ be infinite cyclic. If m and n are integers and $m \neq n$, then $g^m \neq g^n$.

Proof. One of m, n is larger — suppose without loss of generality that $m > n$. I want to show that $g^m \neq g^n$; suppose this is false, so $g^m = g^n$. Then $g^{m-n} = 1$, so g has finite order. This contradicts the fact that a generator of an infinite cyclic group has infinite order. Therefore, $g^m \neq g^n$. □

The next result characterizes subgroups of cyclic groups. The proof uses the Division Algorithm for integers in an important way.

Theorem. Subgroups of cyclic groups are cyclic.

Proof. Let $G = \langle g \rangle$ be a cyclic group, where $g \in G$. Let $H < G$. If $H = \{1\}$, then H is cyclic with generator 1. So assume $H \neq \{1\}$.

To show H is cyclic, I must produce a generator for H. What is a generator? It is an element whose powers make up the group. *A thing should be smaller than things which are "built from" it* — for example, a brick is smaller than a brick building. Since elements of the subgroup are "built from" the generator, the generator should be the "smallest" thing in the subgroup.

What should I mean by "smallest"?

Well, G is cyclic, so everything in G is a power of g. With this discussion as motivation, let m be the smallest positive integer such that $g^m \in H$.

Why is there such an integer m? Well, H contains something other than $1 = g^0$, since $H \neq \{1\}$. That "something other" is either a positive or negative power of g. If H contains a positive power of g, it must contain a *smallest* positive power, by well ordering.

On the other hand, if H contains a negative power of g — say g^{-k}, where $k > 0$ — then $g^k \in H$, since H is closed under inverses. Hence, H again contains positive powers of g, so it contains a *smallest* positive power, by Well Ordering.

So I have g^m, the smallest positive power of g in H. I claim that g^m generates H. I must show that every $h \in H$ is a power of g^k. Well, $h \in H < G$, so at least I can write $h = g^n$ for some n. But by the Division Algorithm, there are unique integers q and r such that

$$n = mq + r, \quad \text{where} \quad 0 \leq r < m.$$

It follows that

$$g^n = g^{mq+r} = (g^m)^q \cdot g^r, \quad \text{so} \quad h = (g^m)^q \cdot g^r, \quad \text{or} \quad g^r = (g^m)^{-q} \cdot h.$$

Now $g^m \in H$, so $(g^m)^{-q} \in H$. Hence, $(g^m)^{-q} \cdot h \in H$, so $g^r \in H$. However, g^m was the *smallest positive power of g lying in H*. Since $g^r \in H$ and $r < m$, the only way out is if $r = 0$. Therefore, $n = qm$, and $h = g^n = (g^m)^q \in \langle g^m \rangle$.

This proves that g^m generates H, so H is cyclic. □

Example. (**Subgroups of the integers**) Describe the subgroups of $\mathbb{Z}$.

Every subgroup of $\mathbb{Z}$ has the form $n\mathbb{Z}$ for $n \in \mathbb{Z}$.

For example, here is the subgroup generated by 13:

$$13\mathbb{Z} = \langle 13 \rangle = \{\ldots - 26, -13, 0, 13, 26, \ldots\}. \quad □$$

Example. Consider the following subset of $\mathbb{Z}$:

$$H = \{30x + 42y + 70z \mid x, y, z \in \mathbb{Z}\}.$$

(a) Prove that H is a subgroup of $\mathbb{Z}$.

(b) Find a generator for H.

(a) First,

$$0 = 30 \cdot 0 + 42 \cdot 0 + 70 \cdot 0 \in H.$$

If $30x + 42y + 70z \in H$, then

$$-(30x + 42y + 70z) = 30(-x) + 42(-y) + 70(-z) \in H.$$

If $30a + 42b + 70c, 30d + 42e + 70f \in H$, then

$$(30a + 42b + 70c) + (30d + 42e + 70f) = 30(a + d) + 42(b + e) + 70(c + f) \in H.$$

Hence, H is a subgroup. □

(b) Note that $2 = (30, 42, 70)$. I'll show that $H = \langle 2 \rangle$.

First, if $30x + 42y + 70z \in H$, then

$$30x + 42y + 70z = 2(15x + 21y + 35z) \in \langle 2 \rangle.$$

Therefore, $H \subset \langle 2 \rangle$.

Conversely, suppose $2n \in \langle 2 \rangle$. I must show $2n \in H$.

The idea is to write 2 as a linear combination of 30, 42, and 70. I'll do this in two steps.

First, note that $(30, 42) = 6$, and

$$30 \cdot 3 + 42 \cdot (-2) = 6.$$

(You can do this by juggling numbers or using the Extended Euclidean algorithm.) Now $(6, 70) = 2$, and

$$6 \cdot 12 + 70 \cdot (-1) = 2.$$

Plugging $6 = 30 \cdot 3 + 42 \cdot (-2)$ into the last equation, I get

$$\begin{aligned} (30 \cdot 3 + 42 \cdot (-2)) \cdot 12 + 70 \cdot (-1) &= 2 \\ 30 \cdot 36 + 42 \cdot (-24) + 70 \cdot (-1) &= 2 \end{aligned}$$

Now multiply the last equation by n:

$$2n = 30 \cdot 36n + 42 \cdot (-24n) + 70 \cdot (-n) \in H.$$

This shows that $\langle 2 \rangle \subset H$.

Therefore, $H = \langle 2 \rangle$. □

Lemma. Let G be a group, and let $g \in G$ have order m. Then $g^n = 1$ if and only if m divides n.

Proof. If m divides n, then $n = mq$ for some q, so $g^n = (g^m)^q = 1$.

Conversely, suppose that $g^n = 1$. By the Division Algorithm,

$$n = mq + r \quad \text{where} \quad 0 \le r < m.$$

Hence,

$$g^n = g^{mq+r} = (g^m)^q g^r \quad \text{so} \quad 1 = g^r.$$

Since m is the smallest positive power of g which equals 1, and since $r < m$, this is only possible if $r = 0$. Therefore, $n = qm$, which means that m divides n. □

Example. (**The order of an element**) Suppose an element g in a group G satisfies $g^{45} = 1$. What are the possible values for the order of g?

The order of g must be a divisor of 45. Thus, the order could be

$$1, \quad 3, \quad 5, \quad 9, \quad 15, \quad \text{or} \quad 45.$$

And the order is certainly not (say) 7, since 7 doesn't divide 45. □

Thus, the order of an element is the *smallest* power which gives the identity the element in two ways. It is *smallest* in the sense of being *numerically* smallest, but it is also *smallest* in the sense that it *divides* any power which gives the identity.

Next, I'll find a formula for the order of an element in a cyclic group.

Proposition. Let $G = \langle g \rangle$ be a cyclic group of order n, and let $m < n$. Then g^m has order $\dfrac{n}{(m,n)}$.

Remark. Note that the order of g^m (the element) is the same as the order of $\langle g^m \rangle$ (the subgroup).

Proof. Since (m,n) divides m, it follows that $\dfrac{m}{(m,n)}$ is an integer. Therefore, n divides $\dfrac{mn}{(m,n)}$, and by the last lemma,

$$(g^m)^{\frac{n}{(m,n)}} = 1.$$

Now suppose that $(g^m)^k = 1$. By the preceding lemma, n divides mk, so

$$\frac{n}{(m,n)} \,\Big|\, k \cdot \frac{m}{(m,n)}.$$

However, $\left(\dfrac{n}{(m,n)}, \dfrac{m}{(m,n)}\right) = 1$, so $\dfrac{n}{(m,n)}$ divides k. Thus, $\dfrac{n}{(m,n)}$ divides any power of g^m which is 1, so it is the order of g^m. □

In terms of $\mathbb{Z}_n$, this result says that $m \in \mathbb{Z}_n$ has order $\dfrac{n}{(m,n)}$.

Example. (**Finding the order of an element**) Find the order of the element a^{32} in the cyclic group $G = \{1, a, a^2, \ldots a^{37}\}$. (Thus, G is cyclic of order 38 with generator a.)

In the notation of the Proposition, $n = 38$ and $m = 32$. Since $(38, 32) = 2$, it follows that a^{32} has order $\dfrac{38}{2} = 19$. □

Example. (**Finding the order of an element**) Find the order of the element $18 \in \mathbb{Z}_{30}$.

In this case, I'm using *additive* notation instead of multiplicative notation. The group is cyclic with order $n = 30$, and the element $18 \in \mathbb{Z}_{30}$ corresponds to a^{18} in the Proposition — so $m = 18$.

$(18, 30) = 6$, so the order of 18 is $\dfrac{30}{6} = 5$. □

Next, I'll give two important Corollaries of the proposition.

Corollary. The generators of $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ are the elements of $\{0, 1, 2, \ldots, n-1\}$ which are relatively prime to n.

Proof. If $m \in \{0, 1, 2, \ldots, n-1\}$ is a generator, its order is n. The Proposition says its order is $\dfrac{n}{(m,n)}$. Therefore, $n = \dfrac{n}{(m,n)}$, so $(m,n) = 1$.

Conversely, if $(m,n) = 1$, then the order of m is

$$\frac{n}{(m,n)} = \frac{n}{1} = n.$$

Therefore, m is a generator of $\mathbb{Z}_n$. □

Example. (**Finding the generators of a cyclic group**) List the generators of:

(a) $\mathbb{Z}_{12}$.

(b) $\mathbb{Z}_p$, where p is prime.

(a) The generators of $\mathbb{Z}_{12}$ are 1, 5, 7, and 11. These are the elements of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ which are relatively prime to 12. □
(b) If p is prime, the generators of $\mathbb{Z}_p$ are 1, 2, ..., $p - 1$. □

Example. (a) List the generators of $\mathbb{Z}_9$.

(b) List the elements of the subgroup $\langle 3 \rangle$ of $\mathbb{Z}_{27}$.

(c) List the generators of the subgroup $\langle 3 \rangle$ of $\mathbb{Z}_{27}$.

(a) The generators are the elements relatively prime to 9, namely 1, 2, 4, 5, 7, and 8. □

(b)

$$\langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}. \quad \Box$$

(c) $\langle 3 \rangle$ is cyclic of order 9, so its generators are the elements corresponding to the generators 1, 2, 4, 5, 7, and 8 of $\mathbb{Z}_9$. Since $27 = 3 \cdot 9$, I can just multiply these generators by 3.

Thus, the generators of $\langle 3 \rangle$ are 3, 6, 12, 15, 21, and 24. □

Corollary. A finite cyclic group of order n contains a subgroup of order m for each positive integer m which divides n.

Proof. Suppose G is a finite cyclic group of order n with generator g, and suppose $m \mid n$. Thus, $mp = n$ for some p.

I claim that g^p generates a subgroup of order m. The preceding proposition says that the order of g^p is $\dfrac{n}{(p, n)}$. However, $p \mid n$, so $(p, n) = p$. Therefore, g^p has order

$$\frac{n}{(p, n)} = \frac{n}{p} = m.$$

In other words, g^p generates a subgroup of order m. □

In fact, it's possible to prove that there is a *unique* a subgroup of order m for each m dividing n.

Note that for an *arbitrary* finite group G, it isn't true that if $n \mid |G|$, then G contains a cyclic subgroup of order n.

Example. (**Subgroups of a cyclic group**) (a) List the subgroups of $\mathbb{Z}_{15}$.

(b) List the subgroups of $\mathbb{Z}_{24}$.

(a) $\mathbb{Z}_{15}$ contains subgroups of order 1, 3, 5, and 15, since these are the divisors of 15. The subgroup of order 1 is the identity, and the subgroup of order 15 is the entire group.

The last result says: If n divides 15, then there is a subgroup of order n — in fact, a *unique* subgroup of order n.

Since $\mathbb{Z}_{15}$ is cyclic, these subgroups must be cyclic. They are generated by 0 and the nonzero elements in $\mathbb{Z}_{15}$ which divide 15: 1, 3, and 5.

Lagrange's theorem (which I'll discuss later) says that in any finite group, the order of a subgroup must divide the order of the group. In this context, Lagrange's theorem says if H is a subgroup of order n, then n divides 15.

Putting these results together, this means that you can find *all* the subgroups of $\mathbb{Z}_{15}$ by taking $\{0\}$ (the trivial subgroup), together with the cyclic subgroups generated by the nonzero elements in $\mathbb{Z}_{15}$ which divide 15: 1, 3, and 5.

1 generates $\mathbb{Z}_{15}$.

5 generates a subgroup of order 3:

$$\langle 5\rangle = \{0, 5, 10\}.$$

3 generates a subgroup of order 5:

$$\langle 3\rangle = \{0, 3, 6, 9, 12\}. \quad \square$$

(b) Since the divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24, the subgroups of $\mathbb{Z}_{24}$ are:

$$\langle 0\rangle, \quad \langle 1\rangle, \quad \langle 2\rangle, \quad \langle 3\rangle, \quad \langle 4\rangle, \quad \langle 6\rangle, \quad \langle 8\rangle, \quad \langle 12\rangle.$$

The subgroup generated by 3 has order 8:

$$\langle 3\rangle = \{0, 3, 6, 9, 12, 15, 18, 21\}. \quad \square$$

Example. (**A product of cyclic groups**) Consider the group

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(m, n) \mid m \in \mathbb{Z}_2, n \in \mathbb{Z}_3\}.$$

Show that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic by finding a generator.

The operation is componentwise addition:

$$(m, n) + (m', n') = (m + m', n + n').$$

It is routine to verify that this is a group, the **direct product** of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

The element $(1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_3$ has order 6:

$$\begin{aligned}(1,1) + (1,1) &= (0,2),\\ (1,1) + (0,2) &= (1,0),\\ (1,1) + (1,0) &= (0,1),\\ (1,1) + (0,1) &= (1,2),\\ (1,1) + (1,2) &= (0,0).\end{aligned}$$

Hence, $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic of order 6. More generally, if $(m, n) = 1$, then $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic of order mn. Be careful! — $\mathbb{Z}_2 \times \mathbb{Z}_2$ is *not* the same as $\mathbb{Z}_4$! $\square$
