Ideals and Subrings

A subgroup of a group is a subset of the group which is a group in its own right, using the operation it inherits from its parent group. Likewise, a **subring** of a ring is a subset of the ring which is a ring in its own right, using the addition and multiplication it inherits from its parent ring.

Definition. Let R be a ring. A **subring** is a subset $S \subset R$ such that:

(a) S is closed under addition: If $a, b \in S$, then $a + b \in S$.

(b) The zero element of R is in S: $0 \in S$.

(c) S is closed under additive inverses: If $a \in S$, then $-a \in S$.

(d) S is closed under multiplication: If $a, b \in S$, then $ab \in S$.

It turns out to be useful to consider certain other kinds of "subobjects" of rings: **Ideals**. I'll use ideals to construct **quotient rings**, which just as I used normal subgroups to construct quotient groups.

Definition. Let R be a ring. An **ideal** S of R is a subset $S \subset R$ such that:

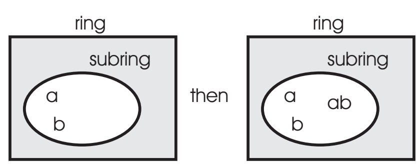
(a) S is closed under addition: If $a, b \in S$, then $a + b \in S$.

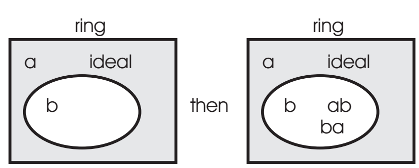
(b) The zero element of R is in S: $0 \in S$.

(c) S is closed under additive inverses: If $a \in S$, then $-a \in S$.

(d) If $r \in R$ and $x \in S$, then $rx \in S$ and $xr \in S$. In other words, S is closed under multiplication (on either side) by arbitrary ring elements.

What's the difference between a subring and an ideal? A subring must be closed under multiplication of elements *in the subring*. An ideal must be closed under multiplication of an element in the ideal by *any* element in the ring.





Since the ideal definition requires *more* multiplicative closure than the subring definition, every ideal is a subring. The converse is false, as I'll show by example below.

In the course of attempting to prove Fermat's Last Theorem, mathematicians were led to introduce rings in which **unique factorization** failed — that is, it might be possible to factor a ring element into primes in more than one way. They were led to introduce *ideal numbers* (essentially what are now called *ideals*) in an attempt to restore unique factorization.

What I've defined above is usually called a **two-sided ideal**. If I only require that $rx \in S$ for $r \in R$ and $x \in S$, I get **left ideals**. Likewise, if I only require that $xr \in S$ for $r \in R$ and $x \in S$, I get **right ideals**.

From now on, if I just say "ideal", I will mean a two-sided ideal.

If R is commutative, then $rb = br$, so you only need to check that one of rb, br, is in S. In the commutative case, there's no difference between left ideals, right ideals, and two-sided ideals.

Lemma. Let R be a ring. Then R and $\{0\}$ are ideals.

Proof. R is a group under addition, and as such I've already proved that R (the whole group) and $\{0\}$ (the set consisting of the identity) are subgroups of R. Thus, they are both closed under addition, contain 0, and are closed under taking additive inverses. I only have to verify the fourth ideal axiom in each case.

For R, if $x \in R$ and $r \in R$, then $xr, rx \in R$, because R is closed under multiplication (being the whole ring!). Therefore, R is an ideal.

For $\{0\}$, take $0 \in \{0\}$ — what other choice do you have? — and $r \in R$. Then

$$r \cdot 0 = 0 \in \{0\} \quad \text{and} \quad 0 \cdot r = 0 \in \{0\}.$$

Therefore, $\{0\}$ is an ideal. $\square$

Definition. Let R be a ring. **A proper ideal** is an ideal other than R; a **nontrivial ideal** is an ideal other than $\{0\}$.

Example. (**The integers as a subset of the reals**) Show that $\mathbb{Z}$ is a subring of $\mathbb{R}$, but not an ideal.

$\mathbb{Z}$ is a subring of $\mathbb{R}$: It contains 0, is closed under taking additive inverses, and is closed under addition and multiplication. With regard to multiplication, note that the product of two integers is an integer.

However, $\mathbb{Z}$ is *not* an ideal in $\mathbb{R}$. For example, $\sqrt{2} \in \mathbb{R}$ and $3 \in \mathbb{Z}$, but $\sqrt{2} \cdot 3 \notin \mathbb{Z}$. $\square$

Example. (**An ideal in the ring of integers**) Show that the subset $n\mathbb{Z}$ is an ideal in $\mathbb{Z}$ for $n \in \mathbb{Z}$.

We already know that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ under addition. So I just need to check closure under multiplication.

Let $k \in \mathbb{Z}$ and let $nx \in n\mathbb{Z}$, where $x \in \mathbb{Z}$. Then

$$k \cdot (nx) = n(kx) \in n\mathbb{Z}.$$

Therefore, $n\mathbb{Z}$ is an ideal. $\square$

Example. (**An ideal in a product ring**) In the ring $\mathbb{Z}_4 \times \mathbb{Z}_4$, consider the subset

$$I = \{(0,0), (1,1), (2,2), (3,3)\}.$$

Show that I is a subring, but not an ideal.

It's easy to check that I is a subring of $\mathbb{Z}_4 \times \mathbb{Z}_4$. First, I contains the additive identity $(0,0)$.

Next, a typical element of I has the form (n,n). The additive inverse is

$$-(n,n) = (-n,-n) = (4-n, 4-n) \in I.$$

If you add two elements of I, you get an element of I:

$$(a,a) + (b,b) = (a+b, a+b).$$

(Of course, you'll reduce $a + b \bmod 4$, but the two components remain the same.)

Finally, if you multiply two elements of I, you get an element of I:

$$(a, a)(b, b) = (ab, ab).$$

However, I is not an ideal; for example, $(2, 2) \in I$, but

$$(3, 0) \cdot (2, 2) = (2, 0) \notin I.$$

In other words, I is closed under multiplication of elements *inside* I, but not closed under multiplication by an element from *outside* I. $\Box$

Definition. Let R be a commutative ring, and let $a \in R$. The **principal ideal generated by** a is

$$\langle a \rangle = \{ra \mid r \in R\}.$$

For example, in the ring of polynomials with real coefficients $\mathbb{R}[x]$, this is the principal ideal generated by $x^2 + 4$:

$$\langle x^2 + 4 \rangle = \{(x^2 + 4) \cdot f(x) \mid f(x) \in \mathbb{R}[x]\}.$$

It's the set consisting of all multiples of $x^2 + 4$. For example, here are some elements of $\langle x^2 + 4 \rangle$:

$$(2x + 5) \cdot (x^2 + 4), \quad (-\pi x^{50} + \sqrt{2}) \cdot (x^2 + 4), \quad 0 = 0 \cdot (x^2 + 4).$$

We'd better check that the principal ideal really is an ideal!

Lemma. Let R be a commutative ring, and let $a \in R$. Then $\langle a \rangle$ is a two-sided ideal in R.

Proof. First, $0 = 0 \cdot a \in \langle a \rangle$.

If $ra \in \langle a \rangle$, then $-(ra) = (-r)a \in \langle a \rangle$.

Finally, if $ra, sa \in \langle a \rangle$, then $ra + sa = (r + s)a \in \langle a \rangle$.

Thus, $\langle a \rangle$ is an additive subgroup of R.

If $ra \in \langle a \rangle$ and $s \in R$, then

$$s(ra) = (sr)a \in \langle a \rangle \quad \text{and} \quad (ra)s = (rs)a \in \langle a \rangle.$$

Therefore, $\langle a \rangle$ is a two-sided ideal. $\Box$

Definition. Let $I_1, \ldots, I_n$ be ideals in a ring R. The **ideal sum** is

$$\sum_{k=1}^{n} I_k = \{x_1 + \cdots + x_n \mid x_k \in I_k\}.$$

Definition. Let I and J be ideals in a ring R. The **ideal product** is

$$IJ = \{x_1 y_1 + \cdots + x_n y_n \mid x_i \in I, y_i \in J\}.$$

Thus, IJ consists of all finite sums of products xy, $x \in I$, $y \in J$.

Proposition. Let R be a ring.

(a) Suppose R has an identity and I is an ideal. If $1 \in I$, then $I = R$.

(b) The intersection $I \cap J$ of (left, right, two-sided) ideals I and J is a (left, right, two-sided) ideal.

(c) If $I_1, \ldots, I_n$ are (left, right, two-sided) ideals, the ideal sum is a (left, right, two-sided) ideal.

(d) If I and J are (left, right, two-sided) ideals, the ideal product is a (left, right, two-sided) ideal.

Proof. I'll prove the first statement by way of example. Let I be an ideal in a ring with 1. $I \subset R$, so I need to prove $R \subset I$. Let $r \in R$. Now $1 \in I$, so by the definition of an ideal, $r = r \cdot 1 \in I$. Therefore, $R \subset I$, so $R = I$. $\square$