10-15-2018

Absolute Maxima and Minima

A common problem is to find the *largest* or *smallest* value of something, usually subject to certain conditions. The "something" will be modelled by a function $f(x)$; here is a precise definition of what I mean by "largest" and "smallest".

Suppose c is a point in the domain of a function $f(x)$. Then:

1. An **absolute maximum** occurs at c if $f(x) \leq f(c)$ for all x in the domain of f.

2. An **absolute minimum** occurs at c if $f(x) \geq f(c)$ for all x in the domain of f.

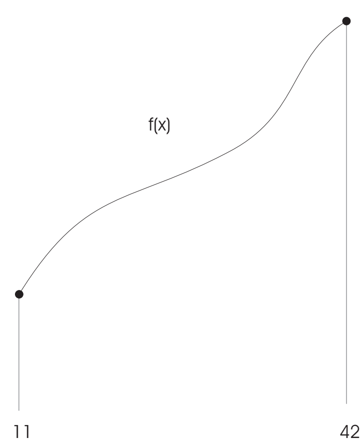
Sometimes it's important to consider points which are only largest or smallest in small parts of a graph.

1. A **relative** (or **local**) **maximum** occurs at c if $f(x) \leq f(c)$ for all x in an open interval containing c.

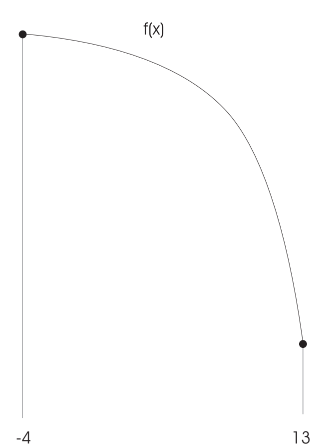
2. A **relative** (or **local**) **minimum** occurs at c if $f(x) \geq f(c)$ for all x in an open interval containing c.

If a local max is like being the toughest guy on your block, an absolute max is like being the toughest guy in the world.

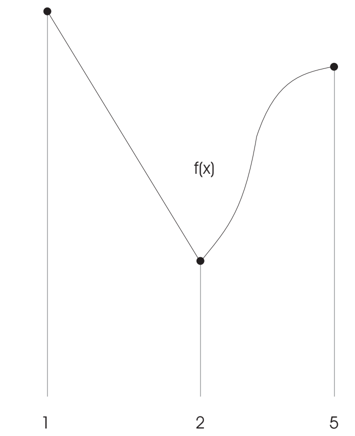
Local maxima and minima are important in graphing functions, among other things. However, today I'll concentrate on absolute maxima and minima. I'll begin by looking at some pictorial examples.



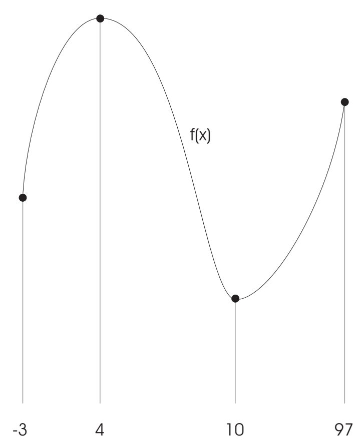
absolute max at $x = 42$
absolute min at $x = 11$



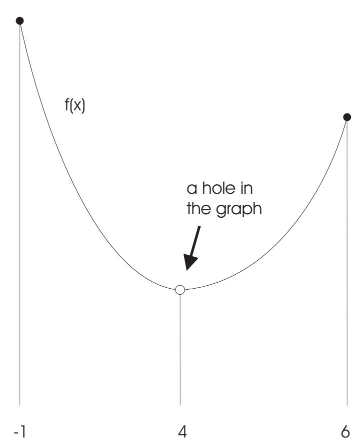
absolute max at $x = -4$
absolute min at $x = 13$



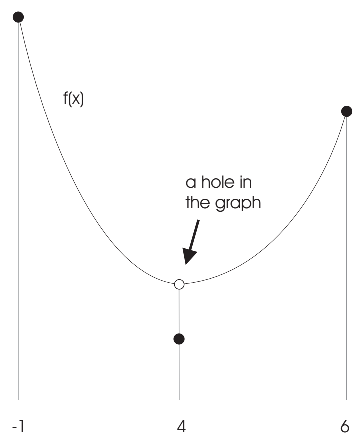
absolute max at $x = 1$
absolute min at $x = 2$
endpoint max at $x = 5$



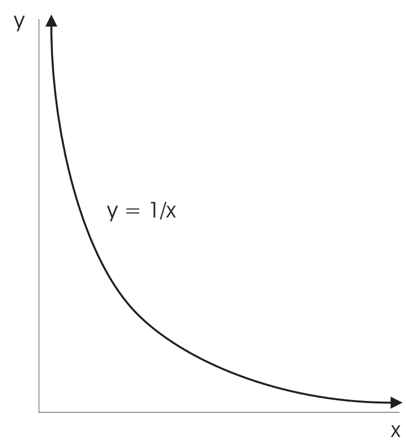
absolute max at $x = 4$
absolute min at $x = 10$
endpoint max at $x = 97$
endpoint min at $x = -3$



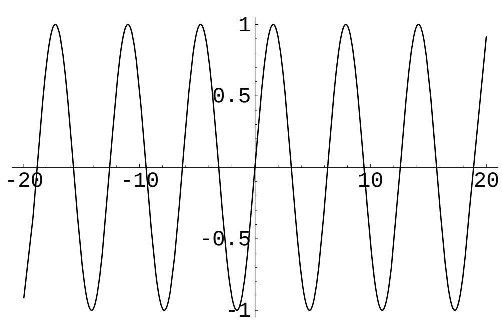
absolute max at $x = -1$
no absolute min
endpoint max at $x = 6$



absolute max at $x = -1$
absolute min at $x = 4$
endpoint max at $x = 6$



no absolute max
no absolute min



infinitely many
absolute maxima and minima

Some conclusions may be drawn from these examples:

(a) A function need not have an absolute max or absolute min. It can have both, one or the other, or neither.

(b) A continuous function *must* have an absolute max and an absolute min on a closed interval.

The second observation is important: It says that if you look for an absolute max or min on a closed interval, at least in principle you won't be disappointed. The proof requires more advanced knowledge about the **topology of the real numbers**, so I'll omit it.

Where can you expect an absolute max or min to occur? Here's a reasonable guess based on the examples above. I'll assume that I'm looking at a function that is continuous on a closed interval, so the secon observation guarantees that I have an absolute max and an absolute min.

Theorem. For a continuous function on a closed interval, an absolute max or min can occur at:

1. A **critical point** for $f(x)$ — that is, a value c in the domain of f where $f'(c)$ is undefined or $f'(c) = 0$.

2. An endpoint of the interval.

Proof. I'll give a sketch of the proof of this result. Assume that f is continuous on an interval $a \le x \le b$ and differentiable on $a < x < b$.

Suppose that c is a max or a min, but c is not an endpoint (a or b) or a place where f' is undefined. I'll show that $f'(c) = 0$.

If $f'(c) \ne 0$, then it's either positive or negative. Assume $f'(c) > 0$ — the argument if $f'(c) < 0$ is similar.

Recall that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that

$$\lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} - f'(c)\right) = 0.$$

Another way of saying this is that I can make $\left|\dfrac{f(x) - f(c)}{x - c} - f'(c)\right|$ as small as I want by making x sufficiently close to c. So make x close enough to c so that

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \frac{1}{2} f'(c).$$

Writing this absolute value inequality as a pair of inequalities, I have

$$\begin{aligned} -\frac{1}{2} f'(c) &< \frac{f(x) - f(c)}{x - c} - f'(c) < \frac{1}{2} f'(c) \\ \frac{1}{2} f'(c) &< \frac{f(x) - f(c)}{x - c} < \frac{3}{2} f'(c) \end{aligned}$$

I got the second set of inequalities by adding $\dfrac{1}{2} f'(c)$ to each term of the first.

Now $f'(c) > 0$, so the last set of inequalities has $\dfrac{f(x) - f(c)}{x - c}$ caught between two positive numbers. Therefore, $\dfrac{f(x) - f(c)}{x - c}$ must be positive. Therefore, the top and bottom of this fraction are either both positive or both negative.

If $x > c$ — that is, x is to the right of c — then $x - c$ is positive, so $f(x) - f(c) > 0$, and $f(x) > f(c)$. In other words, the function f increases as you move from c to the right. Since there are *bigger* values of f to the right of c, it means that c can't be a max.

If $x < c$ — that is, x is to the left of c — then $x - c$ is negative, so $f(x) - f(c) < 0$, and $f(x) < f(c)$. In other words, the function f decreases as you move from c to the left. Since there are *smaller* values of f to the left of c, it means that c can't be a min.

But now I'm stuck, because I assumed that c was either a max or a min.

The only possibility is that my assumption that $f'(c) \ne 0$ was incorrect. So indeed, $f'(c) = 0$. □

This leads to the following procedure for finding the absolute max or min of a function $f(x)$ on a closed interval $a \le x \le b$:

1. Locate the critical points of f *which lie in the interval.*

2. Plug the critical points and the endpoints of the interval into f.

3. The largest values of f correspond to the absolute maxima; the smallest values of f correspond to the absolute minima.

(A remark for people who know some calculus: Do not confuse this with the First or Second Derivative test! You plug the candidate points into $f(x)$, not into $f'(x)$ or $f''(x)$.)

Example. Find the absolute max and absolute min of $f(x) = x^2$ for:

(a) $-1 \le x \le 2$.

(b) $-2 \le x \le 2$.

(c) $1 \le x \le 3$.

(a) First, I'll find the critical points. $f'(x) = 2x$, so $f'(x) = 0$ for $x = 0$. Note that $x = 0$ lies in the interval $-1 \le x \le 2$. There are no values of x for which f is undefined.

The endpoints are $x = -1$ and $x = 2$.

I plug $x = 0$, $x = -1$, and $x = 2$ into $f(x)$:

x	-1	0	2
$f(x)$	1	0	4
		absolute min	absolute max

(b) For $-2 \le x \le 2$, $x = 0$ is the only critical point, and it's in the interval $-2 \le x \le 2$.

The endpoints are $x = -2$ and $x = 2$.

x	-2	0	2
$f(x)$	4	0	4
	absolute max	absolute min	absolute max

Here is a case where two points are "tied" for absolute max.

(c) For $1 \le x \le 3$, $x = 0$ is a critical point, *but it is not in the interval* $1 \le x \le 3$. Therefore, it doesn't count.

The endpoints are $x = 1$ and $x = 3$.

x	1	3
$f(x)$	1	9
	absolute min	absolute max

Let me repeat the warning, since it sometimes trips people up: *If a critical point is not in the interval under consideration, it is not tested.*

Finally, notice that I used the same function $f(x) = x^2$ in these three examples, but the answers were different. *The interval under consideration is important* — it determines which critical points are to be tested, and it contributes its endpoints as candidates. $\square$

Example. Find the absolute maximum and absolute minimum of $f(x) = 2x^3 - 9x^2 - 24x + 2$ on the interval $0 \le x \le 5$.

The derivative is

$$f'(x) = 6x^2 - 18x - 24 = 6(x^2 - 3x - 4) = 6(x-4)(x+1).$$

It's a good idea to write the derivative in factored form, since this makes it easier to read off the critical points.

$f'(x) = 0$ for $x = 4$ and $x = -1$. $f'(x)$ is defined for all x. The only critical point in the interval $0 \le x \le 5$ is $x = 4$. I test the critical point and the endpoints by plugging them into f:

x	0	4	5
$f(x)$	2	-110	-93
	absolute max	absolute min	

The absolute min is -110 and it occurs at $x = 4$; the absolute max is 2 and it occurs at $x = 0$. □

Example. Find the largest and smallest values of

$$f(x) = \frac{9}{2}x^{2/3} - \frac{3}{5}x^{5/3} \quad \text{on the interval} \quad -1 \le x \le 5.$$

I'll do the easy part first: The endpoints are $x = -1$ and $x = 5$.

Next, I'll find the critical points. Compute the derivative:

$$f'(x) = 3x^{-1/3} - x^{2/3}.$$

I would like to simplify $f'(x)$ so that I can read off the critical points. The idea is to *get it to look like one chunk, with everything factored.* You can often accomplish this by:

(a) Writing negative powers as fractions.

(b) Combining fractions over common denominators.

(Some people prefer negative powers, but I think fractions are more visual and easier to work with.)

So

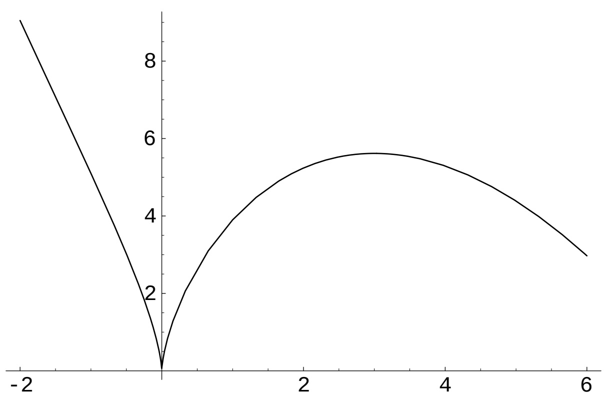
$$f'(x) = 3x^{-1/3} - x^{2/3} = \frac{3}{x^{1/3}} - x^{2/3} = \frac{3}{x^{1/3}} - x^{2/3} \cdot \frac{x^{1/3}}{x^{1/3}} = \frac{3}{x^{1/3}} - \frac{x}{x^{1/3}} = \frac{3-x}{x^{1/3}}.$$

There is nothing to factor, so I'm done.

$f'(x) = 0$ for $x = 3$. And $f'(x)$ is undefined for $x = 0$ (because $x = 0$ would cause division by zero.) Not so fast! — Is f defined at $x = 0$? Checking the original equation, I see that $f(0) = 0$. Since f' is undefined at $x = 0$ *but f is defined at $x = 0$*, $x = 0$ is a critical point.

Note also that both 0 and 3 are in the interval $-1 \le x \le 5$.

You can see the critical points and the endpoints in the graph of the function. Notice that there is a "corner" at $x = 0$, where the derivative is undefined.



Now I plug the critical pints and the endpoints into f:

x	-1	0	3	5
$f(x)$	5.1	0	5.61623	4.38603
		absolute min	absolute max	

Notice that if I'd forgotten to check for places where $f'(x)$ is undefined, I would have missed the absolute min! □

Example. Find the largest and smallest values of

$$f(x) = \frac{3}{7}x^{7/3} - 3x^{1/3} \quad \text{for} \quad -2 \le x \le 8.$$

The endpoints are -2 and 8.
The derivative is

$$f'(x) = x^{4/3} - x^{-2/3}.$$

I write the negative power as a fraction, combine fractions over a common denominator, then factor:

$$f'(x) = x^{4/3} - x^{-2/3} = x^{4/3} - \frac{1}{x^{2/3}} = x^{4/3} \cdot \frac{x^{2/3}}{x^{2/3}} - \frac{1}{x^{2/3}} = \frac{x^2}{x^{2/3}} - \frac{1}{x^{2/3}} = \frac{x^2-1}{x^{2/3}} = \frac{(x-1)(x+1)}{x^{2/3}}.$$

$f'(x) = 0$ for $x = 1$ and for $x = -1$. $f'(x)$ is undefined at $x = 0$; since $f(0)$ is defined, $x = 0$ is a critical point. Since -1, 0, and 1 are in the interval $-2 \le x \le 8$, all of them must be tested.

x	-2	8	-1	0	1
$f(x)$	1.61990	48.85714	2.57143	0	-2.57143

The absolute max is at $x = 8$ and the absolute min is at $x = 1$. □

Example. Let $f(x) = -\frac{1}{x} - \frac{2}{x^2} + \frac{32}{3x^3}$. Find the absolute max and the absolute min of f on the interval $1 \le x \le 6$.

Note that

$$f(x) = -x^{-1} - 2x^{-2} + \frac{32}{3}x^{-3}.$$

So

$$f'(x) = x^{-2} + 4x^{-3} - 32x^{-4} = \frac{1}{x^2} + \frac{4}{x^3} - \frac{32}{x^4} = \frac{1}{x^2} \cdot \frac{x^2}{x^2} + \frac{4}{x^3} \cdot \frac{x}{x} - \frac{32}{x^4} = \frac{x^2+4x-32}{x^4} = \frac{(x+8)(x-4)}{x^4}.$$

$f'(x) = 0$ for $x = -8$ and $x = 4$, but only $x = 4$ is in the interval $1 \le x \le 6$. While $f'(x)$ is undefined at $x = 0$, this is not a critical point since $f(0)$ is undefined — and in any case, it is not in the interval $1 \le x \le 6$.

x	1	4	6
$f(x)$	7.66666...	$-0.20833\ldots$	$-0.17283\ldots$
	max	min	

□
