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Antiderivatives

$F(x)$ is an **antiderivative** of $f(x)$ if

$$\frac{dF(x)}{dx} = f(x).$$

Notation:

$$\int f(x)\,dx = F(x) + C.$$

For example,

$$\int x^3\,dx = \frac{1}{4}x^4 + C, \quad \text{because} \quad \frac{d}{dx}\left(\frac{1}{4}x^4\right) = x^3.$$

In fact, all of the following functions are antiderivatives of x^3, because they all differentiate to x^3:

$$\frac{1}{4}x^4, \quad \frac{1}{4}x^4 + 1, \quad \frac{1}{4}x^4 - 13, \quad \frac{1}{4}x^4 + 157.$$

This is the reason for the "$+C$" in the notation: You can add any *constant* to the "basic" antiderivative $\frac{1}{4}x^4$ and come up with another antiderivative.

C is called the **arbitrary constant**. □

Remark. (a) Antiderivatives are often referred to as **indefinite integrals**, and sometimes I'll refer to $\int f(x)\,dx$ as "the integral of $f(x)$ with respect to x". This terminology is actually a bit misleading, but it's traditional, so I'll often use it. There is another kind of "integral" — the **definite integral** — which is probably more deserving of the name.

(b) The notation "$\int f(x)\,dx$" will also be used for **definite integrals**. The integral sign $\int$ is a stretched-out "S", and comes from the fact that definite integrals are defined in terms of **sums**.

"$\int (\ )\,dx$" is a mathematical object called an **operator**, which roughly speaking is a function which takes functions as inputs and produces functions as outputs. Despite appearances, "dx" isn't a separate thing; in fact, "$\int (\ )\,dx$" is the **whole name** of the antidervative operator. It's a weird name — it consists of three symbols ("$\int$", "d", and "x"), and has a space between the "$\int$" and the "dx" for the input function.

I'll come back to this again when I discuss **substitution**, since at that point this can become a source of confusion.

Every differentiation formula has a corresponding antidifferentiation formula. This makes it easy to derive antidifferentiation rules from the rules for differentiation.

Theorem. (**Power Rule**) For $n \ne -1$,

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

Proof. This follows from the fact that

$$\frac{d}{dx}\frac{1}{n+1}x^{n+1} = x^n.$$

(Notice that the expression on the left is undefined if $n = -1$.) □

Example. Compute the following antiderivatives:

(a) $\int x^{100}\,dx$.

(b) $\int \sqrt{x}\,dx$.

(c) $\int \frac{1}{x^5}\,dx$.

(d) $\int \frac{1}{x^{5/3}}\,dx$.

(a)

$$\int x^{100}\,dx = \frac{1}{101}x^{101} + C.$$

(b)

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C.$$

(c)

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = -\frac{1}{4}x^{-4} + C.$$

(d)

$$\int \frac{1}{x^{5/3}}\,dx = \int x^{-5/3}\,dx = -\frac{3}{2}x^{-2/3} + C. \quad \square$$

Theorem.

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

$$\int k \cdot f(x)\,dx = k \int f(x)\,dx, \quad \text{if } k \text{ is a constant.}$$

$$\int k\,dx = kx + \int f(x)\,dx, \quad \text{if } k \text{ is a constant.}$$

Proof. I'll prove the first formula by way of example; see if you can prove the others.

Suppose that

$$\frac{d}{dx}F(x) = f(x) \quad \text{and} \quad \frac{d}{dx}G(x) = g(x).$$

By definition, this means that

$$\int f(x)\,dx = F(x) + C \quad \text{and} \quad \int g(x)\,dx = G(x) + C.$$

By the rule for the derivative of a sum,

$$\frac{d}{dx}(F(x) + G(x)) = \frac{d}{dx}F(x) + \frac{d}{dx}G(x) = f(x) + g(x).$$

By definition, this means that

$$\int (f(x) + g(x))\,dx = F(x) + G(x) + C. \quad \square$$

Example. Compute the following antiderivatives:

(a) $\int 8x^{10}\,dx$.

(b) $\int \left(3x^4 + 2x + 5\right)\,dx$.

(a)

$$\int 8x^{10}\,dx = 8\int x^{10}\,dx = \frac{8}{11}x^{11} + C.$$

(b)

$$\int \left(3x^4 + 2x + 5\right)\,dx = 3\int x^4\,dx + 2\int x\,dx + 5\int\,dx = \frac{3}{5}x^5 + x^2 + 5x + C. \quad \square$$

Since the derivative of a product is *not* the product of the derivatives, you can't expect that it would work that way for antiderivatives, either.

Example. Compute $\int (x^2-1)(x^4+2)\,dx$.

To do this antiderivative, I *don't* antidifferentiate $x^2 - 1$ and $x^4 + 2$ separately. Instead, I multiply out, then use the rules I discussed above.

$$\int (x^2-1)(x^4+2)\,dx = \int \left(x^6 - x^4 + 2x^2 - 2\right)\,dx = \frac{1}{7}x^7 - \frac{1}{5}x^5 + \frac{2}{3}x^3 - 2x + C. \quad \square$$

Likewise, the derivative of a quotient is *not* the quotient of the derivatives, and it doesn't work that way for antiderivatives.

Example. Compute $\int \frac{x^4+1}{x^2}\,dx$.

Don't antidifferentiate $x^4 + 1$ and x^2 separately! Instead, divide the bottom into the top:

$$\int \frac{x^4+1}{x^2}\,dx = \int \left(x^2 + x^{-2}\right)\,dx = \frac{1}{3}x^3 - x^{-1} + C. \quad \square$$

Every differentiation rule gives an antidifferentiation rule. So

$$\frac{d}{dx}\sin x = \cos x \quad \text{means that} \quad \int \cos x\,dx = \sin x + C.$$

Example. Compute $\int \left(5x^7 + 4\cos x\right)\,dx$.

For example,

$$\int \left(5x^7 + 4\cos x\right)\, dx = \frac{5}{8}x^8 + 4\sin x + C. \quad \square$$

Example. $\dfrac{dy}{dx} = \left(x^2 + \dfrac{1}{x^2}\right)^2$ and $y(1) = \dfrac{1}{5}$. Find y.

To find y, antidifferentiate $\dfrac{dy}{dx}$:

$$y = \int \frac{dy}{dx}\, dx = \int \left(x^2 + \frac{1}{x^2}\right)^2 dx = \int \left(x^4 + 2 + \frac{1}{x^4}\right) dx = \frac{1}{5}x^5 + 2x - \frac{2}{3}\frac{1}{x^3} + C.$$

$y(1) = \dfrac{1}{5}$:

$$\frac{1}{5} = y(1) = \frac{1}{5} + 2 - \frac{2}{3} + C$$

$$C = -\frac{4}{3}$$

Therefore,

$$y = \frac{1}{5}x^5 + 2x - \frac{2}{3}\frac{1}{x^3} - \frac{4}{3}.$$

This process is a simple example of **solving a differential equation with an initial condition**. $\square$

Example. Suppose an object moves with constant acceleration a. Its initial velocity is v_0, and its initial position is s_0. Find its position function $s(t)$.

First, $a(t) = v'(t) = \dfrac{dv}{dt}$, so

$$v = \int a(t)\, dt = \int a\, dt = at + C.$$

When $t = 0$, $v = v_0$, so

$$v_0 = a \cdot 0 + C, \quad C = v_0.$$

Therefore,

$$v = at + v_0.$$

Next, $v(t) = s'(t) = \dfrac{ds}{dt}$, so

$$s = \int v(t)\, dt = \int (at + v_0)\, dt = \frac{1}{2}at^2 + v_0 t + D.$$

When $t = 0$, $s = s_0$:

$$s_0 = \frac{1}{2}a \cdot 0 + v_0 \cdot 0 + D, \quad D = s_0.$$

Therefore,

$$s = \frac{1}{2}at^2 + v_0 t + s_0.$$

For example, an object falling near the surface of the earth experiences a constant acceleration of -32 feet per second per second (negative, since the object's height s is *decreasing*). Its height at time t is

$$s = -16t^2 + v_0 t + s_0.$$

Here v_0 is its initial velocity and s_0 is the height from which it's dropped. $\square$
