10-16-2018

Concavity and the Second Derivative Test

If $y = f(x)$ is a function, the **second derivative** of y (or of f) is the derivative of the first derivative. Notation:

$$\frac{d^2y}{dx^2}, \quad \frac{d^2f}{dx^2}, \quad y'', \quad f''.$$

Thus,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Example. Find the second derivatives of the following functions.

(a) $y = x^2$.

(b) $y = \dfrac{1}{x^2}$.

(c) $y = \dfrac{1}{6}x^3 - 2x^2 + 5x + 4$.

(a)

$$y' = 2x, \quad y'' = 2. \quad \square$$

(b)

$$y' = -\frac{2}{x^3}, \quad y'' = \frac{6}{x^4}. \quad \square$$

(c)

$$y' = \frac{1}{2}x^2 - 4x + 5, \quad y'' = x - 4. \quad \square$$

The first derivative gives information about whether a function increases or decreases. In fact:

(a) A differentiable function increases on intervals where its derivative is positive, and vice versa.

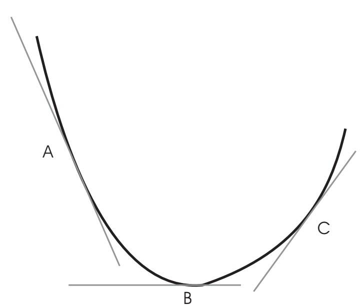
(b) A differentiable function decreases on intervals where its derivative is negative, and vice versa.

A function $y = f(x)$ is **concave up** on an open interval if y'' is positive on the interval. And a function $y = f(x)$ is **concave down** on an open interval if y'' is negative on the interval.

A point where the concavity goes from up to down or from down to up is called an **inflection point**.

What do these conditions mean geometrically?

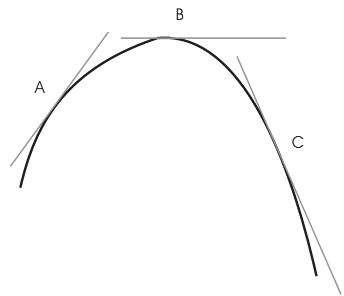
Consider the curve below.



The tangent line A has negative slope, the tangent line B has zero slope, and the tangent line C has positive slope. Therefore, as you move from left to right, the slope of the tangent line *increases*.

But the slope of the tangent line is given by y', and to say something increases means its derivative is positive. So the derivative of y' — which is y'' — must be positive. By the definition, this means the curve is concave up.

Now consider the curve below.

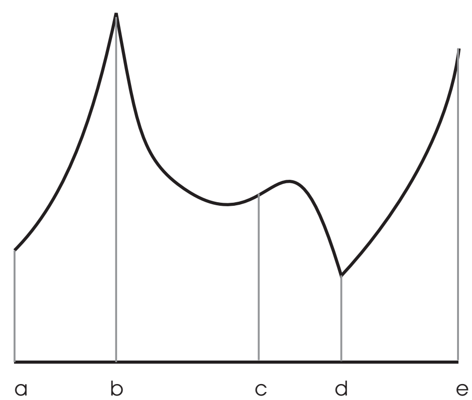


The tangent line at A has positive slope, the tangent line at B has zero slope, and the tangent line at C has negative slope. As you move from left to right, the slope of the tangent line decreases.

The slope of the tangent line is given by y', and to say something decreases means its derivative is negative. So the derivative of y' — which is y'' — must be negative. By the definition, this means the curve is concave down.

The two pictures exemplify the geometric meanings of **concave up** and **concave down**.

Example. The graph of a function is pictured below.



Determine the intervals on which the function is concave up and the intervals on which it is concave down. Find the x-coordinates of any inflection points.

The graph is concave up on $a < x < b$, $b < x < c$, and $d < x < e$. The graph is concave down on $c < x < d$.

Note that concavity is a property of a graph on an open interval, so the endpoints aren't included.

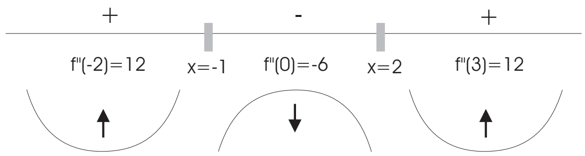
There are inflection points at $x = c$ and at $x = d$. □

Example. Find the intervals on which $y = \frac{1}{4}x^4 + \frac{1}{2}x^3 - 3x^2 + 6$ is concave up and the intervals on which it is concave down. Find the x-coordinates of any inflection points.

$$y' = x^3 + \frac{3}{2}x^2 - 6x, \quad y'' = 3x^2 - 3x - 6 = 3(x^2 - x - 2) = 3(x - 2)(x + 1).$$

I set up a sign chart for y'', just as I use a sign chart for y' to tell where a function increases and where it decreases. The break points for my concavity sign chart will be the x-values where $y'' = 0$ and the x-values where y'' is undefined.

In this case, $y'' = 0$ for $x = 2$ and $x = -1$, and there are no points where y'' is undefined. The break points are at $x = 2$ and $x = -1$.



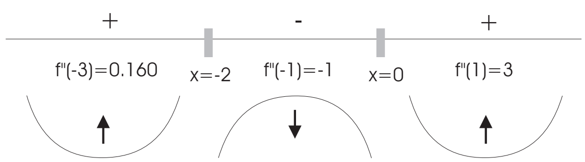
I picked numbers in each interval and plugged the numbers into y''. If y'' is positive, I put a "+" on the interval and draw a concave-up curve below the interval; if y'' is negative, I put a "−" on the interval and draw a concave-down curve below the interval.

The function is concave up for $x < -1$ and for $x > 2$. It is concave down for $-1 < x < 2$. $x = -1$ and $x = 2$ are inflection points. $\square$

Example. Find the intervals on which $y = \dfrac{9}{4}x^{4/3} - 9x^{1/3}$ is concave up and the intervals on which it is concave down. Find the x-coordinates of any inflection points.

$$y' = 3x^{1/3} - 3x^{-2/3}, \quad y'' = x^{-2/3} + 2x^{-5/3} = \frac{x+2}{x^{5/3}}.$$

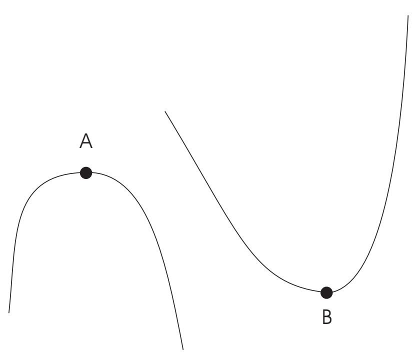
$y'' = 0$ for $x = -2$; y'' is undefined for $x = 0$.



The function is concave up for $x < -2$ and for $x > 0$. It is concave down for $-2 < x < 0$. $x = -2$ and $x = 0$ are inflection points. $\square$

Concavity provides way to tell whether a critical point is a max or a min — well, sometimes. This method is called the **Second Derivative Test**.

Consider a critical point where $y' = 0$, i.e. where the tangent line is horizontal. Here are two possibilities.



The point A is a local max; it occurs at a place where the curve is concave down, i.e. where $y'' < 0$.

The point B is a local min; it occurs at a place where the curve is concave up, i.e. where $y'' > 0$.

Theorem. Suppose f'' is defined on an open interval, and for some point c in the interval $f'(c) = 0$. Then:

(a) If $f''(c) < 0$, then $x = c$ is a local max.

(b) If $f''(c) > 0$, then $x = c$ is a local min.

(c) If $f''(c) = 0$, the test fails. Try the First Derivative Test.

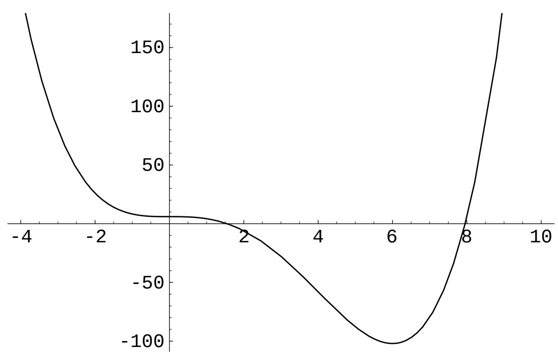
Example. Use the Second Derivative Test to classify the critical points of $y = \frac{1}{4}x^4 - 2x^3 + 6$.

$$y' = x^3 - 6x^2 = x^2(x-6), \quad y'' = 3x^2 - 12x.$$

The critical points are $x = 0$ and $x = 6$.

x	$y'' = 3x^2 - 12x$	Result
6	$36 > 0$	min
0	0	Test fails

Here's the graph:



In fact, $x = 0$ is neither a max nor a min. □

Remark. It is *not* true that if $f'(c) = 0$ (so c is a critical point) and $f''(c) = 0$ (so the Second Derivative Test fails), then $x = c$ is neither a max nor a min. To say the test fails means that you can draw *no conclusion*, and you need to do more work. The point could *still* be a max or a min!

For example, consider $y = x^4$. Then $y' = 4x^3$ and $y'' = 12x^2$, so $y'(0) = 0$ and $y''(0) = 0$. Thus, $x = 0$ is a critical point, and the Second Derivative Test fails. Nevertheless, $x = 0$ is a local min, as you can verify by using the First Derivative Test.

This example also shows that if $y''(c) = 0$, it does not mean that c is an inflection point. In fact, the graph of $y = x^4$ is always concave up, so the concavity does not change at $x = 0$.