8-29-2018

Continuity

In many cases, you can compute $\lim_{x \to a} f(x)$ by plugging a in for x:

$$\lim_{x \to a} f(x) = f(a).$$

For example,

$$\lim_{x \to 3} (2x^3 - 5x + 1) = 2 \cdot 3^3 - 5 \cdot 3 + 1 = 40.$$

This situation arises often enough that it has a name.

Definition. A function $f(x)$ is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a).$$

This definition really comprises three things, each of which you need to check to show that f is continuous at a:

1. $f(a)$ is defined.

2. $\lim_{x \to a} f(x)$ is defined.

3. The two are equal: $\lim_{x \to a} f(x) = f(a)$.

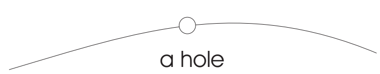
What does this mean geometrically? Here are the three criteria above in pictorial language:

1. "$f(a)$ is defined" means there's a point on the graph at a.

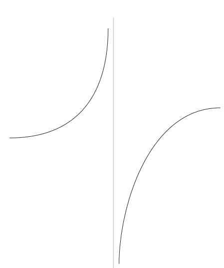
2. "$\lim_{x \to a} f(x)$ is defined" means the graph approaches a single numerical value as you get close to a.

3. "$\lim_{x \to a} f(x) = f(a)$" means that the value you're approaching is the value that f actually takes on — there are "no surprises".

The first criterion means that there can't be a hole or gap in the graph. This also rules out vertical asymptotes. Here are some pictures of these kinds of **discontinuities**:

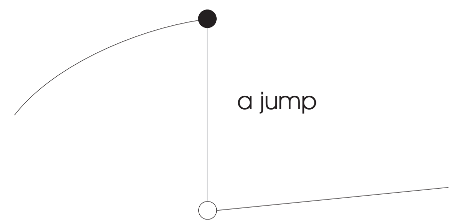


a hole



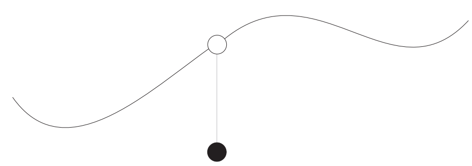
a vertical asymptote

The second criterion means that the graph can't "jump" at a. This is a **jump discontinuity**:



A jump discontinuity occurs when the left and right-hand limits aren't equal.

The third criterion means that the graph is "filled in" at $x = a$ as you'd expect. You *don't* get close to a expecting one value and then find that $f(a)$ is something different, as you do below:



This is called a **removable discontinuity** because you could make the function continuous by filling in the hole. In terms of limits, it means that $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$.

Here are some classes of continuous functions:

(a) A polynomial $p(x)$ is continuous for all x.

(b) $|x|$ is continuous for all x.

(c) Trigonometric functions are continuous wherever they are defined.

(d) e^x is continuous for all x and $\ln x$ is continuous for $x > 0$.

(e) $\sqrt[n]{x}$ is continuous for all x for which it's defined.

The statement about polynomials, for example, follows from a property of limits. If $p(x)$ is a polynomial, I showed that

$$\lim_{x \to c} p(x) = p(c).$$

This is exactly what it means for $p(x)$ to be continuous at c.

For example, $f(x) = 10x^{100} - 15x + 41$ is continuous for all x, since it's a polynomial.

The statement for $|x|$ is a fairly easy limit proof, but I'll omit it. And the statements for trig functions, e^x, and $\ln x$ depend on careful definitions of these functions; I'll discuss some of this later. As an example of the statement about trig functions, $\tan x$ is continuous for all x except odd multiples of $\frac{\pi}{2}$. ($\tan x$ is undefined at odd multiples of $\frac{\pi}{2}$.)

The limit proof for $\sqrt[n]{x}$ is not too hard, but once you have results on e^x and $\ln x$, you can use the fact that

$$\sqrt[n]{x} = x^{1/n} = e^{(\ln x)/n}.$$

You have to ensure that the root you're taking is defined at the point. For example, $\sqrt{x}$ is continuous for $x \geq 0$. Remember that $\sqrt{x}$ is undefined for $x < 0$.

Example. What kind of discontinuity does $f(x) = \dfrac{x^2 - 1}{x^2 - x - 2}$ have at $x = -1$?

Note that
$$\lim_{x \to -1} \frac{x^2 - 1}{x^2 - x - 2} = \lim_{x \to -1} \frac{(x-1)(x+1)}{(x-2)(x+1)} = \lim_{x \to -1} \frac{x-1}{x-2} = \frac{2}{3}.$$

However, $f(-1)$ is undefined. Therefore, there is a removable discontinuity at $x = -1$. I could make the function continuous at $x = -1$ by defining $f(-1) = \dfrac{2}{3}$. □

Example. Let
$$f(x) = \begin{cases} x^2 + 3 & \text{if } x > 2 \\ 4x & \text{if } x \le 2 \end{cases}.$$

What kind of discontinuity does $f(x)$ have at $x = 2$?

Note that
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 3) = 7 \quad \text{and} \quad \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 4x = 8.$$

Thus, the left and right-hand limits aren't equal. Therefore, there is a jump discontinuity at $x = 2$. □

You can also get continuous functions by combining continuous functions in various ways.

Theorem. (a) If f and g are continuous at $x = c$, so is their sum $f + g$.

(b) If f and g are continuous at $x = c$, so is their difference $f - g$.

(c) If f and g are continuous at $x = c$, so is their product fg.

(d) If f and g are continuous at $x = c$, and if $g(c) \ne 0$, then the quotient $\dfrac{f}{g}$ is continuous at $x = c$.

(e) If f is continuous at $g(c)$ and if g is continuous at $x = c$, then the *composite* $f \circ g$ is continuous at $x = c$.

Proof. The proofs are fairly easy consequences of our theorems on limits. I'll prove (c) by way of example.
Suppose f and g are continuous at $x = c$. This means that
$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c).$$

Multiplying the left sides and the right sides, I get
$$\left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right) = f(c)g(c).$$

By the rule for the limit of a product,
$$\lim_{x \to c} f(x)g(x) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right).$$

Therefore,
$$\lim_{x \to c} f(x)g(x) = f(c)g(c).$$

This is what it means for $f(x)g(x)$ to be continuous at $x = c$. □

Here are some illustrations of these rules.

Since e^x and x^3 are continuous for all x, their sum $e^x + x^3$ and their product $x^3 e^x$ are continuous for all x.

The quotient $\dfrac{e^x}{x^3}$ is continuous for all x except $x = 0$ (where the quotient is undefined).

Composition is an important method for constructing continuous functions. For example, $f(x) = \sin x$ is continuous for all x. The polynomial $g(x) = x^4 - 7x^2 + x + 1$ is also continuous for all x. The composite is

$$f(g(x)) = \sin(x^4 - 7x^2 + x + 1).$$

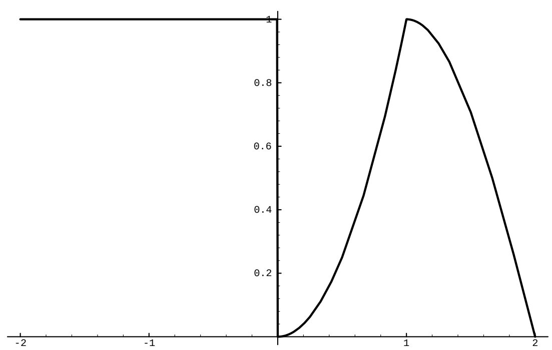
It is continuous for all x.

Example. Let

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 1 \\ \sin \dfrac{\pi x}{2} & \text{if } x \ge 1 \end{cases}.$$

For what values of x is f continuous?

The function is continuous except possibly at the "break points" between the three pieces. I must check the points $x = 0$ and $x = 1$ separately.



At $x = 0$,

$$\lim_{x\to 0-} f(x) = 1 \quad \text{but} \quad \lim_{x\to 0+} f(x) = 0.$$

Since the left- and right-hand limits do not agree,

$$\lim_{x\to 0} f(x) \text{ is undefined.}$$

Hence, f is not continuous at $x = 0$.

At $x = 1$,

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} x^2 = 1, \quad \text{and} \quad \lim_{x\to 1+} f(x) = \lim_{x\to 1+} \sin \frac{\pi x}{2} = 1.$$

The left- and right-hand limits agree, so

$$\lim_{x\to 1} f(x) = 1.$$

Now $f(1) = 1$, so

$$\lim_{x\to 1} f(x) = 1 = f(1).$$

Therefore, f is continuous at $x = 1$.

Conclusion: f is continuous for all x except $x = 0$. □

Example. Let

$$f(x) = \frac{x^2 - 2x - 3}{x^2 - 1}.$$

For what values of x is f continuous?

f is a quotient of two polynomials, and polynomials are continuous for all x. Hence, $f(x)$ is continuous at all points except those which make the bottom equal to 0.

Write f as

$$f(x) = \frac{(x-3)(x+1)}{(x-1)(x+1)}.$$

Hence, f is continuous for all x except $x = 1$ and $x = -1$. (Note that you can't cancel the $x + 1$-terms before seeing where f is undefined.)

However, the discontinuity at $x = -1$ is a removable discontinuity:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - 2x - 3}{x^2 - 1} = \lim_{x \to -1} \frac{(x-3)(x+1)}{(x-1)(x+1)} = \lim_{x \to -1} \frac{x-3}{x-1} = \frac{-4}{-2} = 2.$$

$f(-1)$ is undefined, but if I defined $f(-1) = 2$, then the new f would be continuous at $x = -1$.

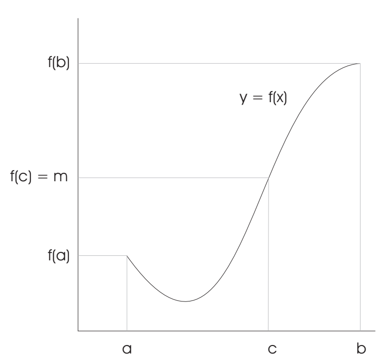
On the other hand, the discontinuity at $x = 1$ is a vertical asymptote; no matter how I define $f(1)$, the function will still be discontinuous at $x = 1$. □

Continuous functions possess the **intermediate value property**. Roughly put, it says that a if continuous function goes from one value to another, it doesn't skip any values in between. This corresponds to the geometric intuition that the graph of a continuous function doesn't have any gaps, jumps, or holes. Here is the precise statement.

Theorem. (**Intermediate Value Theorem**) Let $f(x)$ be a continuous function on the interval $a \le x \le b$. If m is a number between $f(a)$ and $f(b)$, then there is a number c in the interval $a \le x \le b$ such that

$$f(c) = m.$$

The theorem is illustrated in the picture below:



Try it for yourself: Pick any height m between $f(a)$ and $f(b)$. Move horizontally from your chosen height to the graph, then downward from the graph till you hit the x-axis. The place where you hit the x-axis is c. You'll always be able to do this if f is continuous. The intuitive idea is that, being continuous, f can't skip any values in going from $f(a)$ to $f(b)$.

A proof of the Intermediate Value Theorem uses some deep properties of the real numbers, so I won't give it here. At least you can see from the picture that the result is geometrically reasonable.

The theorem illustrates an important point: *You can know something exists without being able to find it.*

If I take your car keys and throw them into a nearby corn field, you *know* that your keys are in the field — but *finding* them is a different story!

The Intermediate Value Theorem says *there is* a number c such that $f(c) = m$. It doesn't tell you how to *find* it, though you can usually *approximate* c as closely as you want.

And by the way — there may be *more than one* number c which works. Even though the statement of the theorem says "there is" (singular), mathematicians use these words to mean "there is at least

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Example. Suppose f is a continuous function, $f(4) = 11$, and $f(7) = 2$. Prove that for some number x between 4 and 7, $f(x) + x^2 = 42$.

Since x^2 and $f(x)$ are continuous, $f(x) + x^2$ is continuous. Plug in 4 and 7:

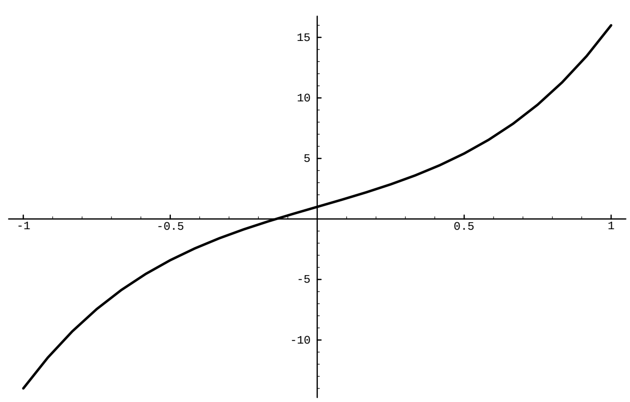
x	$f(x) + x^2$
4	$f(4) + 4^2 = 27$
7	$f(7) + 7^2 = 51$

42 is between 27 and 51, so I can apply the Intermediate Value Theorem to $f(x)+x^2$. It says that there is a number x between 4 and 7 such that $f(x) + x^2 = 42$. □

Example. Approximate a solution to the equation

$$x^5 + 7x^3 + 7x + 1 = 0.$$

Here's the graph:



It looks as though there's a root between -0.5 and 0.

A clever person might say at this point: "Why not just look up the general formula for solving a 5-th degree equation?" After all, there's the general quadratic formula for quadratics ... and there's a general cubic formula and a general quartic formula, though you'd probably have to look them up in a book of tables.

Unfortunately, you'll never find a general quintic formula in any book of tables. Nils Henrik Abel and and Paolo Ruffini showed almost 150 years ago that there's no general quintic, and Evariste Galois showed a little later that you won't have any luck with higher degree equations, either.

You can still *approximate* the root, and the Intermediate Value Theorem *guarantees* that there is one.

f (being a polynomial) is surely continuous. In this situation, the IVT says that f can't go from negative to positive without passing through 0 somewhere in between.

Notice that

$$f(-0.5) \approx -3.40625 \quad \text{and} \quad f(0) = 1.$$

Thus, I know there's a root between -0.5 and 0.

I'll approximate the root by **bisection**. At each step, I'll know the root is caught between two numbers. I'll plug the midpoint into f. The root is now on one side or the other, and I just keep going.

This is exactly what common sense would lead you to do.

Here's the computation:

x	$f(x)$ positive	$f(x)$ negative
-0.5		-3.40625
0.0	1	
-0.25		-0.860352
-0.125	0.111298	
-0.1875		-0.358874
-0.15625		-0.120546

At this point, the root c is caught between -0.125 (the last x which made f positive) and -0.15625 (the last x which made f negative). These two numbers are 0.03125 apart. Hence, the midpoint $x = -0.140625$ is within $0.03125/2 = 0.015625$ of the actual root. The estimate $x = -0.140625$ is therefore good to within 1 or 2 one-hundredths. □