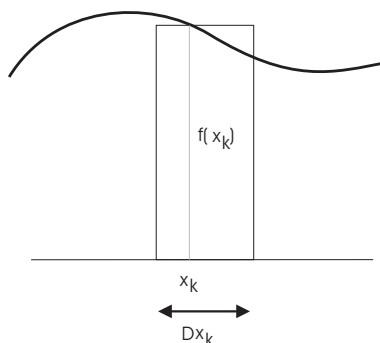


Definite Integrals

We've seen that we can *approximate* the area under a curve by using a sum of rectangle areas. But suppose I'm trying to find the *exact* area under a curve — say $y = f(x)$ from $x = a$ to $x = b$. Since increasing the number of rectangles increases the accuracy of the approximation, maybe I'd get the exact area if I could use an “infinite number of rectangles”. I will give a rough description of how we might do this.

To do this, I start by breaking the interval $[a, b]$ up into n subintervals of lengths $\Delta x_1, \Delta x_2, \dots, \Delta x_n$. In the k -th subinterval, I pick some point x_k , and use $f(x_k)$ as the height of the k -th rectangle.



The sum of the areas of the rectangles approximates the area under the curve:

$$\text{Area} \approx \sum_{k=1}^n f(x_k) \Delta x_k.$$

The more rectangles I take, the better the approximation. So it's reasonable to suppose that the *exact* area would be given by the *limit* of such sums, as n goes to infinity:

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x_k.$$

The expression on the right — a limit of a sum, or a **Riemann sum** — is called **the definite integral of $f(x)$ from a to b** and is denoted as follows:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x_k.$$

If the limit of the sum on the right side is defined, the function f is **integrable** on the interval $a \leq x \leq b$.

Remark. In the definition above, I'm passing over a number of technical points for the sake of getting the intuitive idea across. For example, what does it mean to let the number of rectangles become infinite? Does it matter how the rectangles' widths or the evaluation points are chosen?

A careful treatment of the **Riemann integral** has to take these considerations into account.

It is possible to compute areas using the formula above, though it's not easy. Here's an example of a direct computation using the definition.

Example. Use the limit of a sum to compute the area under $y = 2x + 3$ from $x = 0$ to $x = 2$ — that is, the definite integral

$$\int_0^2 (2x + 3) dx.$$

I will need the following summation formulas:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^n c = nc.$$

Divide up the interval $[0, 2]$ into n equal subintervals. Each has length $\Delta x = \frac{2}{n}$. I'll evaluate the function at the right-hand endpoints. These are

$$1 \cdot \frac{2}{n}, 2 \cdot \frac{2}{n}, 3 \cdot \frac{2}{n}, \dots, (n-1) \cdot \frac{2}{n}, n \cdot \frac{2}{n}.$$

The k -th point is $\frac{2k}{n}$, so the height of the k -th rectangle is

$$f\left(\frac{2k}{n}\right) = \frac{4k}{n} + 3.$$

The sum of the rectangles' areas is

$$\sum_{k=1}^n \left(\frac{4k}{n} + 3\right) \cdot \frac{2}{n} = \frac{2}{n} \sum_{k=1}^n \left(\frac{4}{n} \cdot k + 3\right) = \frac{2}{n} \left(\frac{4}{n} \cdot \sum_{k=1}^n k + \sum_{k=1}^n 3\right).$$

Applying the sum formulas above, I get

$$\frac{2}{n} \left(\frac{4}{n} \cdot \frac{n(n+1)}{2} + 3n\right) = \frac{2}{n} (2(n+1) + 3n) = \frac{2}{n} (5n+2) = \frac{10n+4}{n}.$$

The (exact) area is

$$\lim_{n \rightarrow \infty} \frac{10n+4}{n} = 10. \quad \square$$

While this approach works, it's pretty complicated! I'll discuss better ways to compute definite integrals shortly.

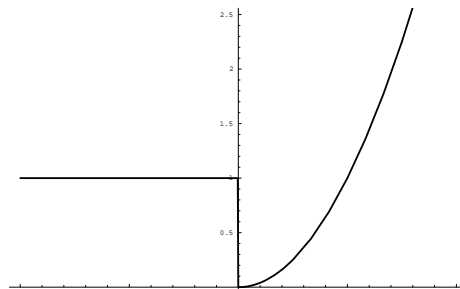
Not all functions are integrable. The following result which I'll state without proof covers most of the cases we'll need:

Theorem. A bounded function with finitely many discontinuities is integrable. \square

For example, consider the function

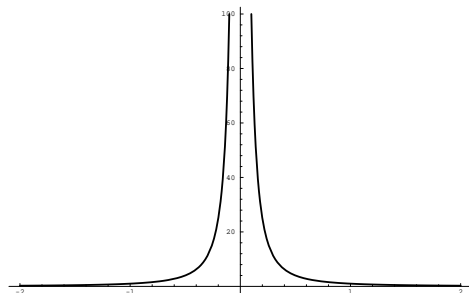
$$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}.$$

There is a single discontinuity at $x = 0$. Hence, f is integrable on any interval.



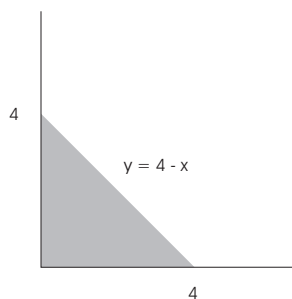
In particular, a continuous function is integrable.

On the other hand, $f(x) = \frac{1}{x^2}$ is not bounded near $x = 0$, so it is not integrable on any interval containing 0.



In some cases, you can use the fact that the definite integral represents the area under a curve to evaluate the integral geometrically.

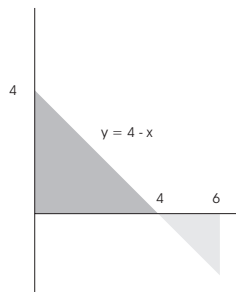
Example. Compute $\int_0^4 (4 - x) dx$.



The integral represents the area of a triangle with height 4 and base 4, so

$$\int_0^4 (4 - x) dx = \frac{1}{2} \cdot 4 \cdot 4 = 8. \quad \square$$

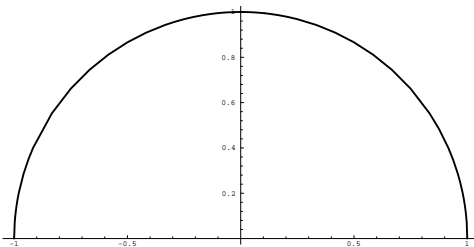
Example. Compute $\int_0^6 (4 - x) dx$.



The area consists of the piece in the last problem, together with a piece of area 2. But the second piece is below the x -axis, so it is taken as *negative* in the definite integral:

$$\int_0^6 (4 - x) dx = 8 - 2 = 6. \quad \square$$

Example. Compute $\int_{-1}^1 \sqrt{1-x^2} dx$.



This is half the area of a circle of radius 1:

$$\int_{-1}^1 \sqrt{1-x^2} dx = \frac{\pi}{2} = 1.57079\dots \quad \square$$

Here are some properties of definite integrals. I'll present them without proofs.

1. If k is a number, then

$$\int_a^b k dx = k(b-a).$$

This is another way of saying that the area of a rectangle is the base times the height.



Example. Compute $\int_5^{13} 7 dx$.

$$\int_5^{13} 7 dx = 7 \cdot (13 - 5) = 56. \quad \square$$

2.

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

This says that the integral of a sum is the sum of the integrals.

3. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x) dx \geq \int_a^b g(x) dx.$$

This says that bigger functions have bigger integrals.

Example. Use the inequality $1 \geq \frac{x^2}{x^2+1}$ to estimate $\int_2^7 \frac{x^2}{x^2+1} dx$.

$$\int_2^7 1 dx \geq \int_2^7 \frac{x^2}{x^2+1} dx, \quad \text{or} \quad 5 \geq \int_2^7 \frac{x^2}{x^2+1} dx. \quad \square$$

4.

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

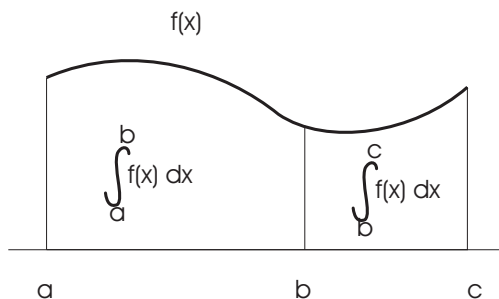
That is, switching the limits of integration multiplies the integral by -1 .
As a special case,

$$\int_a^a f(x) dx = 0.$$

5.

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx.$$

That is, integrating from a to b and then from b to c is the same as integrating all the way from a to c :



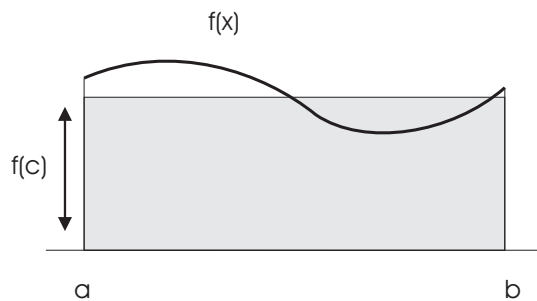
Example. Simplify $\int_3^4 f(x) dx - \int_2^{-1} f(x) dx - \int_3^2 f(x) dx$.

$$\int_3^4 f(x) dx - \int_2^{-1} f(x) dx - \int_3^2 f(x) dx = \int_3^4 f(x) dx + \int_{-1}^2 f(x) dx + \int_2^3 f(x) dx = \int_{-1}^4 f(x) dx. \quad \square$$

6. (**The Mean Value Theorem for Integrals**) There is a number c , $a \leq c \leq b$, such that

$$\int_a^b f(x) dx = f(c) \cdot (b - a).$$

$f(c)$ represents the height of a rectangle on the integral $[a, b]$ which has the same area as the area under the curve.



Example. Use the Mean Value Theorem for Integrals to estimate $\int_0^2 \frac{dx}{x^2 + 1}$.

Applying the Mean Value Theorem for Integrals, I find that there is a number c between 0 and 2 such that

$$\int_0^2 \frac{dx}{x^2 + 1} = \frac{1}{c^2 + 1} \cdot (2 - 0) = \frac{2}{c^2 + 1}.$$

Now

$$0 \leq c \leq 2 \quad \text{gives} \quad 0 \leq c^2 \leq 4.$$

Therefore,

$$1 \leq c^2 + 1 \leq 5$$

$$\frac{1}{1} \geq \frac{1}{c^2 + 1} \geq \frac{1}{5}$$

$$2 \geq \frac{2}{c^2 + 1} \geq \frac{2}{5}$$

So

$$2 \geq \int_0^2 \frac{dx}{x^2 + 1} \geq \frac{2}{5}.$$
