

Differentiation Rules

While it's possible to compute derivatives using the definition of the derivative as a limit, it is too much work if you're differentiating anything complicated. It's more efficient to develop a set of **differentiation rules** to use as shortcuts.

Before we start, here are several comments. First, it is silly to *memorize* the rules. You should learn them by doing examples.

Second, computing derivatives is a *skill* — it is something you learn by practicing. You can't learn to differentiate by watching the instructor do examples. Since you will need to compute derivatives in virtually all the calculus courses you take, it is worth learning to do well now to avoid grief in the future.

On the other hand, you should keep things in perspective. Calculus is *not* just a matter of computing derivatives and integrals. The computations acquire their meaning when they are used to do things: Compute rates of change, find maxima and minima, determine areas or volumes, and so on.

The fact that calculators and computers can do arithmetic does not mean that people don't need to learn arithmetic. A certain amount of computation is necessary in order to develop an intuition for numbers.

In the same way, the fact that computer programs can compute derivatives and integrals does not mean that people don't need to learn to differentiate or integrate. I don't think you can have a complete understanding of what a derivative *is* if you've never computed a few derivatives. However, at some point you may want to use a computer to do the ugly computations, in the same way that you use a calculator to do arithmetic.

I'll illustrate the differentiation rules with some examples. In these examples, I'll write “ $\frac{d}{dx}$ (junk)” for the derivative of junk with respect to x .

1. (**Derivative of a Constant**) The derivative of a constant is 0.

This makes sense, because a constant doesn't change, and the derivative gives the rate of change. Thus,

$$\frac{d}{dx} 17^{-3} = 0.$$

$$\frac{d}{dx} \pi = 0.$$

$$\frac{d}{dt} [(\sin t)^2 + (\cos t)^2] = \frac{d}{dt} 1 = 0.$$

Proofs of the derivative rules depend on the properties we proved for limits. I'll give proofs of some of the rules by way of example.

Proof. (The Derivative of a Constant) In this case, $f(x) = c$, where c is a constant. This means that the output of f is c , *no matter what the input is*. Hence, $f(x+h) = c$ as well. Then

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0. \quad \square$$

2. (**The Power Rule**) $\frac{d}{dx} x^n = nx^{n-1}$, if n is a nonzero integer.

This rule is called the **Power Rule**, because it tells you how to differentiate a power of the variable. In words, you “bring the power down” and decrease the power by 1.

Example.

$$\frac{d}{dx} x^{217} = 217x^{216}.$$

$$\frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$$\frac{d}{dx} \frac{1}{x^7} = \frac{d}{dx} x^{-7} = -7x^{-8} = \frac{-7}{x^8}.$$

Notice that n can be any nonzero real number:

$$\frac{d}{dx} x^{\sqrt{2}} = \sqrt{2} x^{\sqrt{2}-1}. \quad \square$$

Proof. (Power Rule) I'll just consider the case where n is a positive integer. the proof will use the **Binomial Formula**, which I'll give below. Let $f(x) = x^n$. Then

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}.$$

If $n = 1$, this comes out to

$$\lim_{h \rightarrow 0} \frac{(x+h) - x}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$$

This is the same as $1 \cdot x^0$, so the formula works. So assume that $n \geq 2$. Recall that $n!$ (n **factorial**) is the product of the integers from 1 to n :

$$n! = 1 \cdot 2 \cdot \dots \cdot n.$$

The binomial formula says

$$(x+h)^n = x^n + \frac{n!}{1!(n-1)!} x^{n-1} h + \frac{n!}{2!(n-2)!} x^{n-2} h^2 + \frac{n!}{3!(n-3)!} x^{n-3} h^3 + \dots + \frac{n!}{(n-1)!1!} x^1 h^{n-1} + h^n.$$

Therefore,

$$(x+h)^n - x^n = \frac{n!}{1!(n-1)!} x^{n-1} h + \frac{n!}{2!(n-2)!} x^{n-2} h^2 + \frac{n!}{3!(n-3)!} x^{n-3} h^3 + \dots + \frac{n!}{(n-1)!1!} x^1 h^{n-1} + h^n.$$

Next, divide by h :

$$\frac{(x+h)^n - x^n}{h} = \frac{n!}{1!(n-1)!} x^{n-1} + \frac{n!}{2!(n-2)!} x^{n-2} h + \frac{n!}{3!(n-3)!} x^{n-3} h^2 + \dots + \frac{n!}{(n-1)!1!} x^1 h^{n-2} + h^{n-1}.$$

As $h \rightarrow 0$, all the terms but the first go to 0, because they contain positive powers of h . Therefore,

$$f'(x) = \frac{n!}{1!(n-1)!} x^{n-1} = n x^{n-1}.$$

I simplified the fraction by writing

$$\frac{n!}{1!(n-1)!} = \frac{1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n}{1 \cdot 2 \cdot \dots \cdot (n-1)} = n.$$

If n is a negative integer, you can use the result for positive powers together with the **Quotient Rule**. For more general powers, the easiest way to prove the Power rule is to express x^n as $e^{n \ln x}$. This requires knowing the differentiation rules for e^x and $\ln x$, as well as the Chain Rule. \square

3. (**Multiplication by Constants**) $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x).$

This rule says that constants may be “pulled out” when you differentiate. Another way to think of this is that the constant just “sits there”, while the thing it multiplies gets differentiated.

Don’t confuse this rule with the first rule — the derivative of a constant is 0. In that situation, the constant was “by itself”. In this situation, the constant multiplies something.

Example. Compute $\frac{d}{dx}8x^{10}$.

By the Power Rule and the Multiplication by Constants rule,

$$\frac{d}{dx}8x^{10} = 8\frac{d}{dx}x^{10} = 8 \cdot 10x^9 = 80x^9.$$

With some experience, most people will just multiply the 8 and the 10 in their heads and write “80” in one step. \square

Example. I’ll discuss trig functions later on, and I’ll show that

$$\frac{d}{dx}\sin x = \cos x.$$

Compute $\frac{d}{dx}7\sin x$.

$$\frac{d}{dx}7\sin x = 7\cos x. \quad \square$$

The **second derivative** of f is the derivative of $\frac{df}{dx}$ with respect to x . It’s denoted $\frac{d^2f}{dx^2}$. Thus,

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right).$$

Likewise, the third derivative is the derivative of the second derivative, and so on:

$$\frac{d^3f}{dx^3} = \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right).$$

Example. Compute the first, second, and third derivatives of $f(x) = x^{100}$.

$$\frac{df}{dx} = 100x^{99}, \quad \frac{d^2f}{dx^2} = 99 \cdot 100x^{98}, \quad \frac{d^3f}{dx^3} = 98 \cdot 99 \cdot 100x^{97}. \quad \square$$

Definition. A function $f(x)$ has a **vertical tangent** at a if f is continuous at a and

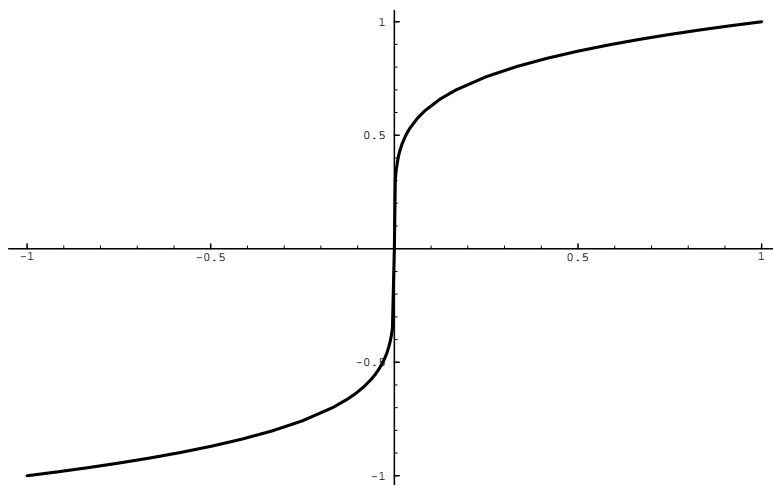
$$\lim_{x \rightarrow a} |f'(x)| = +\infty.$$

Example. Show that $f(x) = x^{1/5}$ has a vertical tangent at $x = 0$.

I have $f'(x) = \frac{1}{5}x^{-4/5}$, so

$$\lim_{x \rightarrow 0} |f'(x)| = \lim_{x \rightarrow 0} \frac{1}{5}x^{-4/5} = \frac{1}{5} \lim_{x \rightarrow 0} \frac{1}{x^{4/5}} = +\infty.$$

Thus, the graph has a vertical tangent at $x = 0$. Picture:



□

Definition. A function $f(x)$ has a **cusp** at a if f is continuous at a and:

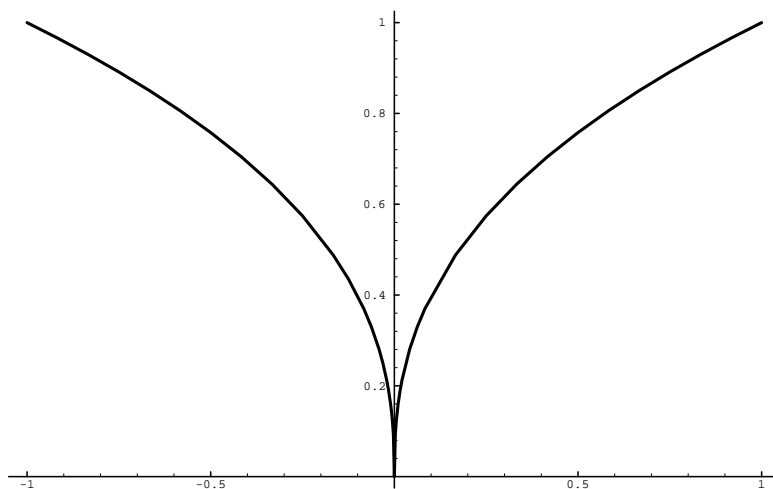
- (a) $f'(x)$ approaches $+\infty$ from one side of a .
- (b) $f'(x)$ approaches $-\infty$ from the other side of a .

Example. Show that $f(x) = x^{2/5}$ has a cusp at $x = 0$.

I have $f'(x) = \frac{2}{5}x^{-3/5}$, so

$$\lim_{x \rightarrow 0^+} f'(x) = +\infty \quad \text{while} \quad \lim_{x \rightarrow 0^-} f'(x) = -\infty.$$

Thus, $f(x)$ has a cusp at $x = 0$. Here's the graph:



□

4. (**Sums**) The derivative of a sum is the sum of the derivatives:

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Example. Compute $\frac{d}{dx} \left(9\sqrt{x} - \frac{14}{x^{3/2}} \right)$.

It's a good idea to write \sqrt{x} as $x^{1/2}$ (and in general, to write $\sqrt[n]{x}$ as $x^{1/n}$). That way, you can see the number power to apply the Power Rule.

$$\frac{d}{dx} \left(9\sqrt{x} - \frac{14}{x^{3/2}} \right) = \frac{d}{dx} \left(9x^{1/2} - 14x^{-3/2} \right) = \frac{9}{2}x^{-1/2} + 21x^{-5/2} = \frac{9}{2} \cdot \frac{1}{\sqrt{x}} + \frac{21}{x^{5/2}}. \quad \square$$

5. (The Product Rule)

$$\frac{d}{dx}(u \cdot v) = u \frac{d}{dx}v + v \frac{d}{dx}u.$$

The formula above is the traditional way this rule is stated. However, if you are trying to remember this rule, it's useful to remember it in a way that makes it easy to *recite*, and so that *the statement tells you what to do*. Here's what I'd suggest.

I'll write “*D*” to mean “the derivative of”. Then the rule is

$$D(\text{first}) \cdot (\text{second}) = (\text{first}) \cdot D(\text{second}) + (\text{second})D(\text{first}).$$

This makes it easy to *recite*, and it comes out like: “The first times the derivative of the second plus the second times the derivative of the first.” — or maybe: “first-D-second-plus-second-D-first” for short.

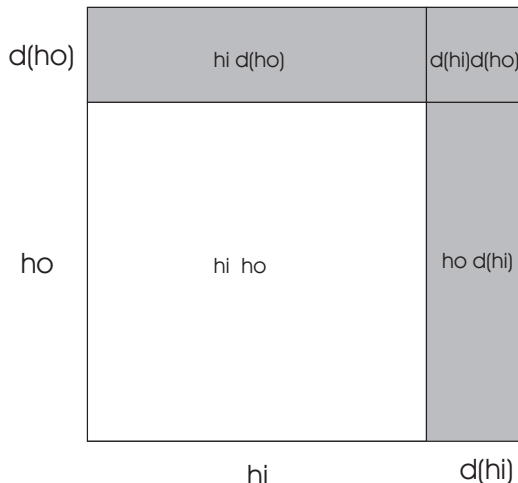
The point is that by using “first” and “second” instead of random symbols like “*u*” and “*v*” (a popular choice), the statement of the rule *tells you what to do*.

Example. Compute $\frac{d}{dx}(x^3 + 2x^2 + 7)(x^4 - x^3 - x + 2)$.

Note: In problems like this, you don't need to multiply answers out unless you're asked to.

$$\frac{d}{dx}(x^3 + 2x^2 + 7)(x^4 - x^3 - x + 2) = (3x^2 + 4x)(x^4 - x^3 - x + 2) + (x^3 + 2x^2 + 7)(4x^3 - 3x^2 - 1). \quad \square$$

The following picture shows why the Product Rule works.



If I change hi by a small amount $d(\text{hi})$ and ho by a small amount $d(\text{ho})$ the change in the product $\text{hi} \cdot \text{ho}$ is

$$d(\text{hi} \cdot \text{ho}) = \text{hi} \cdot d(\text{ho}) + \text{ho} \cdot d(\text{hi}) + d(\text{hi}) \cdot d(\text{ho}).$$

However, the last term is a product of two small numbers, which is *very* small compared to the other two terms. I may neglect it, and I obtain

$$d(\text{hi} \cdot \text{ho}) \approx \text{hi} \cdot d(\text{ho}) + \text{ho} \cdot d(\text{hi}).$$

Now divide by dx to pass to rates of change. Here's the formal proof using limits.

Proof. (Product Rule)

$$\begin{aligned} \frac{d}{dx} [f(x)g(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \\ \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} &= \\ \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \rightarrow 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} &= \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + \lim_{h \rightarrow 0} f(x) \cdot \frac{g(x+h) - g(x)}{h} &= \\ \left(\frac{d}{dx} f(x) \right) \cdot g(x) + f(x) \cdot \left(\frac{d}{dx} g(x) \right). &\quad \square \end{aligned}$$

I'll give the next rule in a form that makes it easy to *recite*, as I did with the Product Rule.

6. (The Quotient Rule)

$$\frac{d}{dx} \frac{\text{top}}{\text{bottom}} = \frac{\text{bottom} \cdot \frac{d}{dx}(\text{top}) - \text{top} \cdot \frac{d}{dx}(\text{bottom})}{\text{bottom}^2}.$$

Some people prefer "hi" and "lo", Take your pick.

Example. (a) Compute $\frac{d}{dx} \frac{x^2 + 1}{x^2 - 3x - 1}$.

(b) Compute $\frac{d}{dx} \frac{x^2 - 4x + 6}{2}$.

(a)

$$\frac{d}{dx} \frac{x^2 + 1}{x^2 - 3x - 1} = \frac{(x^2 - 3x - 1)(2x) - (x^2 + 1)(2x - 3)}{(x^2 - 3x - 1)^2}. \quad \square$$

(b) If either the top or the bottom is just a *number*, it is better *not* to use the Quotient Rule. If the number is on the bottom, you can divide:

$$\frac{d}{dx} \frac{x^2 - 4x + 6}{2} = \frac{d}{dx} \left(\frac{1}{2}x^2 - 2x + 3 \right) = x - 2. \quad \square$$

I'll note now that if the number is on the top, you can use

$$\frac{d}{dx} \frac{\text{number}}{\text{junk}} = - \frac{\text{number} \cdot \frac{d}{dx} \text{junk}}{\text{junk}^2}.$$

You shouldn't memorize this formula; it's a special case of the **Chain Rule**, which I'll talk about later. But in case you want to try it, here's an example:

$$\frac{d}{dx} \frac{5}{x^2 + 3} = -\frac{5 \cdot (2x)}{(x^2 + 3)^2}.$$

Sometimes it's useful to express the differentiation rules using the "prime" notation. Here are the rules for sums, products, and quotients:

(a) (Sums) $(f + g)'(x) = f'(x) + g'(x)$.

(b) (Products) $(fg)'(x) = f(x)g'(x) + g(x)f'(x)$

(c) (Quotients) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.

Example. Suppose f and g are differentiable functions satisfying

$$f(3) = 5, \quad g(3) = 7, \quad f'(3) = 10, \quad g'(3) = -2.$$

(a) Find $(fg)'(3)$.

(b) Find $\left(\frac{f}{g}\right)'(3)$.

(a)

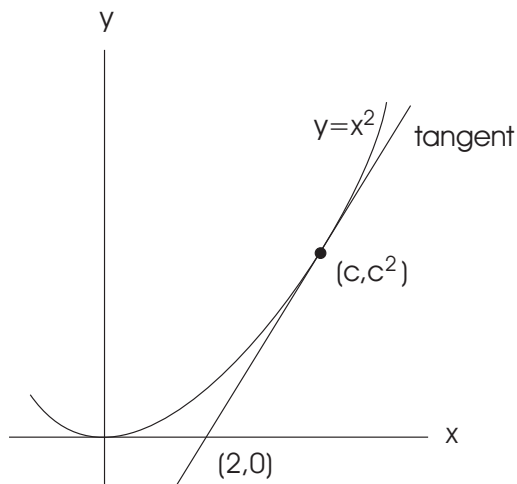
$$(fg)'(3) = f(3)g'(3) + g(3)f'(3) = (5)(-2) + (7)(10) = 60. \quad \square$$

(b)

$$\left(\frac{f}{g}\right)'(3) = \frac{g(3)f'(3) - f(3)g'(3)}{g(3)^2} = \frac{(7)(10) - (5)(-2)}{7^2} = \frac{80}{49}. \quad \square$$

I obtained the derivative by trying to find the slope of the tangent line to a curve. Here are some examples which apply the differentiation rules to problems on tangent lines.

Example. Find the equation of the tangent lines to $y = x^2$ which pass through the point $(2, 0)$.



The x -axis is clearly one such tangent line. Its equation is $y = 0$.

To find the other tangent, let (c, c^2) be the point of tangency. The slope of the tangent line is $y' = 2x$; when $x = c$, $y' = 2c$.

On the other hand, the line passes through $(2, 0)$ and (c, c^2) , so its slope is $\frac{c^2 - 0}{c - 2}$. equate the two expressions for the slope and solve for c :

$$\begin{aligned}2c &= \frac{c^2}{c - 2} \\2c^2 - 4c &= c^2 \\c^2 - 4c &= 0 \\c = 0 \quad \text{or} \quad c &= 4\end{aligned}$$

$c = 0$ gives the x -axis, which I know about already. $c = 4$ gives $y' = 8$; since the line passes through $(2, 0)$, its equation is

$$y = 8(x - 2). \quad \square$$

Example. At what point(s) does the graph of $y = \frac{x}{(x - 1)^2}$ have a horizontal tangent?

A horizontal line has slope 0. Since the derivative gives the slope of the tangent, I'll find the derivative and set it equal to 0:

$$y' = \frac{(x - 1)^2(1) - (x)(2)(x - 1)}{(x - 1)^4} = \frac{-(x + 1)}{(x - 1)^3}.$$

Clearly, $y' = 0$ when $x = -1$. $y(-1) = -\frac{1}{4}$, so there is a horizontal tangent at the point $\left(-1, -\frac{1}{4}\right)$. \square

Example. Find the equation of the line which is *perpendicular* to the graph of $y = x^2 - 3x - 4$ at the point $(3, -4)$.

To be perpendicular to the graph is the same as being perpendicular to the tangent line. The slope of the tangent is given by the derivative, which is $y' = 2x - 3$. Hence, at $x = 3$, the tangent has slope $y'(3) = 3$.

Perpendicular line have slopes which are negative reciprocals of each other. Therefore, the slope of the perpendicular line is $-\frac{1}{3}$. Since the line passes through the point $(3, -4)$, its equation is

$$-\frac{1}{3}(x - 3) = y + 4, \quad \text{or} \quad -\frac{1}{3}x - 3 = y. \quad \square$$