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The Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** says, roughly, that the following processes undo each other:

$$\left\{\begin{matrix} \text{finding slopes} \\ \text{of tangent lines} \end{matrix}\right\} \qquad\qquad \left\{\begin{matrix} \text{finding areas} \\ \text{by rectangle sums} \end{matrix}\right\}$$

The first process is differentiation, and the second process is (definite) integration. To say that the two undo each other means that if you start with a function, do one, then do the other, you get the function you started with.

In equation form, you can say

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where} \quad F(x) \quad \text{is an antiderivative of} \quad f(x).$$

This equation is the key to evaluating definite integrals. It says that if I can find an antiderivative $F(x)$ for $f(x)$, then I can compute the definite integral $\int_a^b f(x)\,dx$ by plugging the limits a and b into $F(x)$.

Another way of putting it is: Finding slopes of tangents and finding rectangle sums should be related in this way.

Theorem. (**The Fundamental Theorem of Calculus (first version)**) Suppose f is integrable on $a \le x \le b$, and that $F'(x) = f(x)$ for some differentiable function F defined on $a \le x \le b$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The Fundamental Theorem of Calculus says that I can compute the definite integral of a function f by finding an antiderivative F of f.

Example. Compute $\int_0^3 x^2\,dx$.

$$\int_0^3 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^3 = 9 - 0 = 9. \quad \square$$

Example. Compute $\int_0^{\pi/2} \cos x\,dx$.

$$\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1.$$

But note that

$$\int_0^{\pi} \cos x\,dx = [\sin x]_0^{\pi} = \sin\pi - \sin 0 = 0.$$

And

$$\int_{\pi/2}^{\pi} \cos x\,dx = [\sin x]_{\pi/2}^{\pi} = \sin\pi - \sin\frac{\pi}{2} = -1.$$

Definite integrals may be positive, negative, or 0. $\square$

Example. Compute $\int_0^2 (3x^5 - 7x + 4)\,dx$.

$$\int_0^2 (3x^5 - 7x + 4)\,dx = \left[\frac{1}{2}x^6 - \frac{7}{2}x^2 + 4x\right]_0^2 = (32 - 7 + 8) - (0 - 0 + 0) = 33. \quad \square$$

Example. Compute $\int_1^2 \frac{x^3 + 1}{x^2}\,dx$.

$$\int_1^2 \frac{x^3 + 1}{x^2}\,dx = \int_1^2 \left(x + \frac{1}{x^2}\right)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{x}\right]_1^2 = \left(2 - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right) = 2. \quad \square$$

If you do an integral using a substitution, you can either use the substitution to change the limits of integration, or put the original variable back at the end.

Example. Compute $\int_0^1 (x^2 + 1)^{10} x\,dx$.

$$\int_0^1 (x^2 + 1)^{10} x\,dx = \int_1^2 u^{10} x \cdot \frac{du}{2x} = \frac{1}{2}\int_1^2 u^{10}\,du =$$

$$\left[u = x^2 + 1, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}; \quad x = 0, u = 1; \quad x = 1, u = 2\right]$$

$$\frac{1}{2}\left[\frac{1}{11}u^{11}\right]_1^2 = \frac{2047}{22}.$$

Alternatively,

$$\int_0^1 (x^2 + 1)^{10} x\,dx = \int_?^? u^{10} x \cdot \frac{du}{2x} = \frac{1}{2}\int_?^? u^{10}\,du =$$

$$\left[u = x^2 + 1, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}\right]$$

$$\frac{1}{2}\left[\frac{1}{11}u^{11}\right]_?^? = \frac{1}{2}\left[\frac{1}{11}(x^2 + 1)^{11}\right]_0^1 = \frac{2047}{22}. \quad \square$$

Example. Compute $\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$.

$$\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int_1^2 \frac{\sin u}{\sqrt{x}} \cdot 2\sqrt{x}\,du = 2\int_1^2 \sin u\,du =$$

$$\left[u = \sqrt{x}, \quad du = \frac{dx}{2\sqrt{x}}, \quad dx = 2\sqrt{x}\,du; \quad x = 1, u = 1; \quad x = 4, u = 2\right]$$

$$2\left[-\cos u\right]_1^2 = 2(\cos 1 - \cos 2) = 1.91289\ldots. \quad \square$$

If the velocity of a particle at time t is $v(t)$, the **change in position** from $t = a$ to $t = b$ is

$$s(b) - s(a) = \int_a^b v(t)\,dt.$$

Example. A particle's velocity is

$$v(t) = 12t^3 + 2t + 1.$$

Find the change in position from $t = 1$ to $t = 3$.

$$\int_1^3 v(t)\,dt = \int_1^3 (12t^3 + 2t + 1)\,dt = \left[3t^4 + t^2 + t\right]_1^3 = 250. \quad \square$$

There is another version of the Fundamental Theorem which says in a direct way that "integration and differentation are opposites".

Theorem. (**Fundamental Theorem, Second Version**) Suppose f is continuous on an interval $a \le x \le b$. Then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This says that if you start with a function ("$f(t)$"), integrate ("$\int_a^x \cdot\,dt$"), then differentiate ("$\dfrac{d}{dx}$"), you get what you started with ("$f(x)$"). This is another way of saying that differentiation and integration are opposite processes.

Proof. I'll prove the second version of the Fundamental Theorem using the first version.

By the definition of the derivative,

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right).$$

Using properties of definite integrals, I can swap the limits on the second integral, then combine the two integrals into one:

$$\lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Suppose that $F(x)$ is an antiderivative of $f(x)$, so $\dfrac{d}{dx}F(x) = f(x)$. Applying the first version of the Fundamental Theorem, I get

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0}\frac{1}{h}\left[F(x)\right]_x^{x+h} = \lim_{h\to 0}\frac{1}{h}\left(F(x+h) - F(x)\right).$$

However, the last expression is just the limit definition of the derivative of $F(x)$. Since $\dfrac{d}{dx}F(x) = f(x)$, I get

$$\lim_{h\to 0}\frac{1}{h}\left(F(x+h) - F(x)\right) = \frac{d}{dx}F(x) = f(x).$$

Putting all the equalities together, I have

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \quad \square$$

Example. (a) Compute $\frac{d}{dx}\int_3^x \sin(t^2)\,dt$.

(b) Compute $\frac{d}{dx}\int_3^{x^5} \sin(t^2)\,dt$.

(a)

$$\frac{d}{dx}\int_3^x \sin(t^2)\,dt = \sin(x^2).$$

Note that the 3 is irrelevant; the answer would be the same if 3 was replaced by (say) 42. □

(b) I can't apply the theorem as is, because the thing I'm differentiating with respect to ("x") doesn't match the upper limit of the integral ("x^5"). Hence, I must apply the Chain Rule first:

$$\frac{d}{dx}\int_3^{x^5} \sin(t^2)\,dt = \frac{dx^5}{dx}\frac{d}{dx^5}\int_3^{x^5} \sin(t^2)\,dt = 5x^4 \cdot \sin\left((x^5)^2\right) = 5x^4\sin(x^{10}).$$

Notice that in applying the Chain Rule, I got the thing I was differentiating with respect to ("x^5") to match the upper limit of the integral ("x^5"). □

