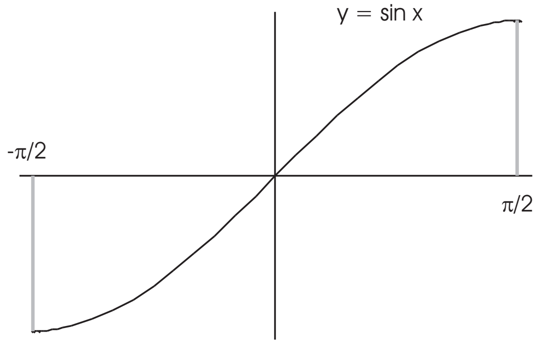
# Inverse Trig Functions

If you restrict $f(x) = \sin x$ to the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, the function *increases*:




This implies that the function is *one-to-one*, and hence it has an inverse. The inverse is called the **inverse sine** or **arcsine function**, and is denoted $\arcsin x$ or $\sin^{-1}(x)$. Note that in the second case $\sin^{-1}(x)$ *does not mean* "$\frac{1}{\sin x}$"!

Note: "Arcsine" (and $\arcsin x$) are older terms, and there is similar terminology for the other inverse trig functions (so "arctangent" and $\arctan x$ for the inverse tangent function, and so on). I'll use the inverse function terminology instead.

In word, $y = \sin^{-1} x$ is *the angle whose sine is* $x$. Another way of saying this is:

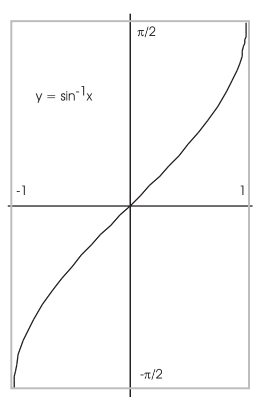
$$y = \sin^{-1} x \quad \text{is the same as} \quad \sin y = x.$$

The fact that sin and $\sin^{-1}$ are inverse functions can be expressed by the following equations:

$$\sin \sin^{-1} a = a \quad \text{for} \quad -1 \leq a \leq 1,$$

$$\sin^{-1} \sin b = b \quad \text{for} \quad -\frac{\pi}{2} \leq b \leq \frac{\pi}{2}.$$

Since the restricted sin takes angles in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and produces numbers in the range $-1 \leq y \leq 1$, $\sin^{-1}$ takes numbers in the range $-1 \leq y \leq 1$ and produces angles in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.



**Example.** Compute $\sin^{-1} \dfrac{1}{2}$ and $\sin^{-1}(-1)$.

$$\sin^{-1} \frac{1}{2} = \frac{\pi}{6}, \quad \text{since} \quad \sin \frac{\pi}{6} = \frac{1}{2}.$$
$$\sin^{-1}(-1) = -\frac{\pi}{2}, \quad \text{since} \quad \sin\left(-\frac{\pi}{2}\right) = -1.$$

Sine and arcsine are inverses, so they undo one another — but you have to be careful!

$$\sin\left(\arcsin \frac{2}{5}\right) = \frac{2}{5}, \quad \text{but} \quad \arcsin(\sin 2\pi) = 0, \quad \text{not} \quad 2\pi.$$

$\sin^{-1}(\text{stuff})$ can't be $2\pi$, because $\sin^{-1}$ always returns an angle in the range $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$. $\square$

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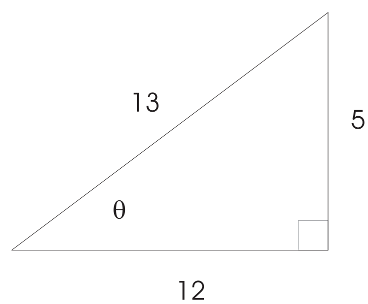
**Example.** Find $\tan \sin^{-1} \dfrac{5}{13}$.

First, let $\theta = \sin^{-1} \dfrac{5}{13}$. This means that $\sin\theta = \dfrac{5}{13}$. Now $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$, so I get the following picture:




I got the adjacent side using Pythagoras: $\sqrt{13^2 - 5^2} = 12$.
Using the triangle, I have

$$\tan \sin^{-1} \frac{5}{13} = \tan\theta = \frac{5}{12}. \quad \square$$

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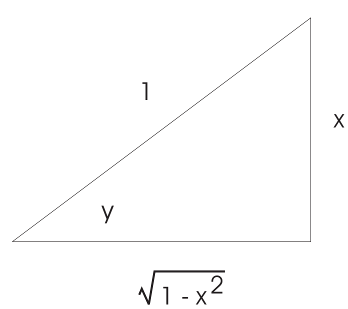
You can find a derivative formula for $\sin^{-1}$ using implicit differentiation. Let $y = \sin^{-1} x$. This is equivalent to $x = \sin y$. Differentiate implicitly:

$$\begin{aligned} x &= \sin y \\ 1 &= (\cos y) y' \\ y' &= \frac{1}{\cos y} \end{aligned}$$

I'd like to express the result in terms of $x$. Here's the right triangle that says $x = \sin y$:



I found the other leg using Pythagoras. You can see that $\cos y = \sqrt{1-x^2}$. Hence, $y' = \dfrac{1}{\sqrt{1-x^2}}$. That is,

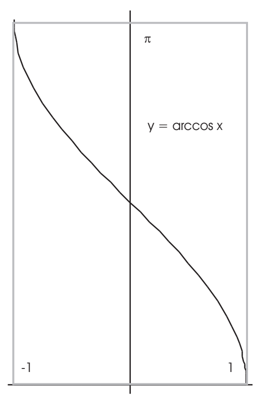
$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

Every derivative formula gives rise to a corresponding antiderivative formula:

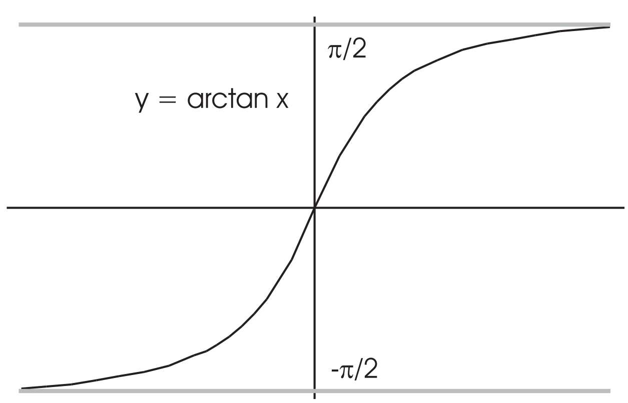
$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C.$$

Before I do some calculus examples, I want to mention some of the other inverse trig functions. I'll discuss the **inverse cosine**, **inverse tangent**, and **inverse secant** functions.

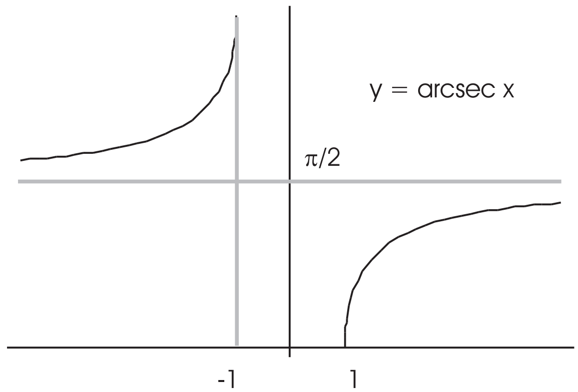
(a) You get the inverse cosine by inverting $\cos x$, restricted to $0 \le x \le \pi$.




(b) You get the inverse tangent by inverting $\tan x$, restricted to $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.



(c) You get the inverse secant by inverting $\sec x$, restricted to $0 < x < \frac{\pi}{2}$ together with $\frac{\pi}{2} < x < \pi$.




As with sin and $\sin^{-1}$, the domains and ranges of these functions and their inverses are "swapped":

| Function | Domain | Range |
|---|---|---|
| $\sin^{-1}$ | $-1 \le x \le 1$ | $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ |
| $\cos^{-1}$ | $-1 \le x \le 1$ | $0 \le x \le \pi$ |
| $\tan^{-1}$ | $-\infty < x < \infty$ | $-\frac{\pi}{2} < x < \frac{\pi}{2}$ |
| $\sec^{-1}$ | $x \le -1, x \ge 1$ | $0 \le x < \frac{\pi}{2}, \frac{\pi}{2} < x \le \pi$ |

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**Example.** Compute $\tan^{-1} 1$ and $\cos^{-1}\left(-\frac{1}{2}\right)$.

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \text{since} \quad \tan\frac{\pi}{4} = 1.$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \quad \text{since} \quad \cos\frac{2\pi}{3} = -\frac{1}{2}. \quad \Box$$

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You can derive the derivative formulas for the other inverse trig functions using implicit differentiation, just as I did for the inverse sine function.

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$

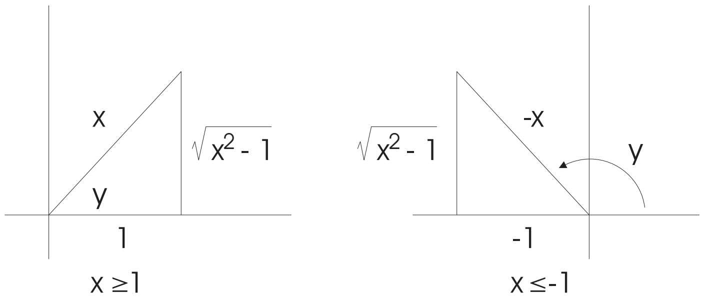
For example, I'll derive the formula for $\frac{d}{dx}\sec^{-1} x$.

The derivation starts out like the derivation for $\dfrac{d}{dx}\sin^{-1} x$. Let $y = \sec^{-1} x$, so $\sec y = x$. Differentiating implicitly, I get

$$(\sec y \tan y)y' = 1$$

$$y' = \frac{1}{\sec y \tan y}$$

There are two cases, depending on whether $x \ge 1$ or $x \le -1$.




Suppose $x \ge 1$. Then $y = \sec^{-1} x$ is in the interval $0 \le y < \dfrac{\pi}{2}$, as illustrated in the first diagram above. You can see from the picture that

$$\sec y = x \quad \text{and} \quad \tan y = \sqrt{x^2 - 1}.$$

Therefore,

$$y' = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2 - 1}}.$$

$x \ge 1$, so $x$ is positive, and $x = |x|$. Therefore,

$$y' = \frac{1}{x\sqrt{x^2 - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Now suppose that $x \le -1$. Then $y = \sec^{-1} x$ is in the interval $\dfrac{\pi}{2} < y \le \pi$, as illustrated in the second diagram above. Since $x$ is negative, the hypotenuse must be $-x$, since it must be positive and since $\sec y = \dfrac{\text{(hypotenuse)}}{\text{(adjacent)}}$ must equal $x$. In this case,

$$\sec y = x \quad \text{and} \quad \tan y = -\sqrt{x^2 - 1}.$$

Therefore,

$$y' = \frac{1}{\sec y \tan y} = \frac{1}{-x\sqrt{x^2 - 1}}.$$

$x \le -1$, so $x$ is negative, and $-x = |x|$. Therefore,

$$y' = \frac{1}{-x\sqrt{x^2 - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

This proves that $y' = \dfrac{1}{|x|\sqrt{x^2 - 1}}$ in all cases.

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**Example.** Compute:

(a) $\dfrac{d}{dx}\left(\sin^{-1}\sqrt{x}+\sqrt{\sin^{-1}x}\right)$.

(b) $\dfrac{d}{dx}\dfrac{1}{\tan^{-1}x}$.

(c) $\dfrac{d}{dx}\sec^{-1}(e^x)$.

(a)

$$\frac{d}{dx}\left(\sin^{-1}\sqrt{x}+\sqrt{\sin^{-1}x}\right)=\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}+\frac{1}{2}\left(\sin^{-1}x\right)^{-1/2}\cdot\frac{1}{\sqrt{1-x^2}}.\quad\square$$

(b)

$$\frac{d}{dx}\frac{1}{\tan^{-1}x}=\left(-\frac{1}{(\tan^{-1}x)^2}\right)\left(\frac{1}{1+x^2}\right).\quad\square$$

(c)

$$\frac{d}{dx}\sec^{-1}(e^x)=\frac{e^x}{e^x\sqrt{e^{2x}-1}}=\frac{1}{\sqrt{e^{2x}-1}}.\quad\square$$

I don't need absolute values in the last example, because $e^x$ is always positive.

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**Example.** Prove the identity

$$\tan^{-1}w+\tan^{-1}\frac{1}{w}=\frac{\pi}{2}.$$

$$\frac{d}{dx}\tan^{-1}\frac{1}{w}=\frac{-\dfrac{1}{w^2}}{1+\dfrac{1}{w^2}}=-\frac{1}{1+w^2}.$$

Hence,

$$\frac{d}{dx}\left(\tan^{-1}w+\tan^{-1}\frac{1}{w}\right)=0.$$

A function with zero derivative is constant, so

$$\tan^{-1}w+\tan^{-1}\frac{1}{w}=C,\quad\text{a constant.}$$

But when $w=1$,

$$C=\tan^{-1}w+\tan^{-1}\frac{1}{w}=\tan^{-1}1+\tan^{-1}1=\frac{\pi}{2}.$$

Therefore,

$$\tan^{-1}w+\tan^{-1}\frac{1}{w}=\frac{\pi}{2}.\quad\square$$

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Here are the integration formulas for some of the inverse trig functions. I'm giving extended versions of the formulas — with "$a^2$" replacing the "1" that you'd get if you just reversed the derivative formulas — in order to save you a little time in doing problems.

$$\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\frac{x}{a}+C$$

$$\int\frac{1}{a^2+x^2}\,dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

$$\int \frac{1}{|x|\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

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For instance, here's how to derive the extended $\sin^{-1}$ integral formula from the formula $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$ using substitution:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \frac{1}{a}\int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}}\,dx = \frac{1}{a}\int \frac{1}{\sqrt{1-u^2}}\cdot a\,du = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}\frac{x}{a} + C.$$

$$\left[u = \frac{x}{a}, \quad du = \frac{dx}{a}, \quad dx = a\,du\right]$$

**Example.** Compute $\int \frac{dx}{4+x^2}$ and $\int \frac{1}{\sqrt{3-x^2}}\,dx$.

Using the $\tan^{-1}$ formula with $a = 2$,

$$\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2} + C.$$

Using the $\sin^{-1}$ formula with $a = \sqrt{3}$,

$$\int \frac{1}{\sqrt{3-x^2}}\,dx = \sin^{-1}\frac{x}{\sqrt{3}} + C. \quad \square$$

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**Example.** Compute $\int \frac{dx}{1+4x^2}$.

$$\int \frac{dx}{1+4x^2} = \int \frac{dx}{1+(2x)^2} = \int \frac{1}{1+u^2}\cdot\frac{du}{2} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(2x) + C.$$

$$\left[u = 2x, \quad du = 2\,dx, \quad dx = \frac{du}{2}\right] \quad \square$$

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**Example.** Compute $\int \frac{x^4\,dx}{1+x^{10}}$.

$$\int \frac{x^4\,dx}{1+x^{10}} = \int \frac{x^4\,dx}{1+(x^5)^2} = \int \frac{x^4}{1+u^2}\cdot\frac{du}{5x^4} = \frac{1}{5}\int \frac{du}{1+u^2} =$$

$$\left[u = x^5, \quad du = 5x^4\,dx, \quad dx = \frac{du}{5x^4}\right]$$

$$\frac{1}{5}\tan^{-1} u + C = \frac{1}{5}\tan^{-1}(x^5) + C. \quad \square$$

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**Example.** Compute $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$.

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-u^2}} \cdot \frac{du}{e^x} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} e^x + C.$$

$$\left[u = e^x, \quad du = e^x\,dx, \quad dx = \frac{du}{e^x}\right] \quad \square$$

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**Example.** Compute $\displaystyle\int \frac{(\sec x)^2\,dx}{\sqrt{1-(\tan x)^2}}$.

$$\int \frac{(\sec x)^2\,dx}{\sqrt{1-(\tan x)^2}} = \int \frac{(\sec x)^2}{\sqrt{1-u^2}} \cdot \frac{du}{(\sec x)^2} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} \tan x + C.$$

$$\left[u = \tan x, \quad du = (\sec x)^2\,dx, \quad dx = \frac{du}{(\sec x)^2}\right] \quad \square$$

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