8-29-2018

Left and Right-Hand Limits

In some cases, you let x approach the number a from the left or the right, rather than "both sides at once" as usual.

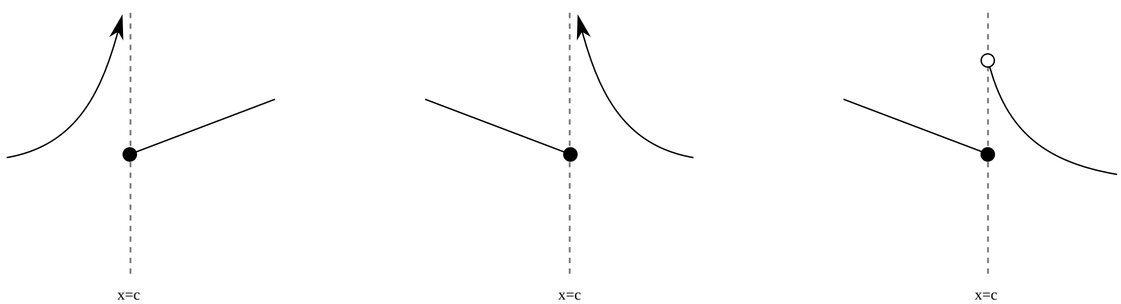
1. $\displaystyle\lim_{x\to c+} f(x)$ means: Compute the limit of $f(x)$ as x approaches c from the *right* — that is, through numbers *bigger than* c.

2. $\displaystyle\lim_{x\to c-} f(x)$ means: Compute the limit of $f(x)$ as x approaches c from the *left* — that is, through numbers *smaller than* c.

These situations may occur if $f(x)$ is only defined to the left or to the right of c. For example, the function $f(x) = \sqrt{x}$ is only defined for $x \ge 0$ (because the square root of a negative number is not a real number).

It's also possible to consider left and right-hand limits when $f(x)$ is defined on both sides of c. In this case, the important question is: Are the left and right-hand limits equal?

Example. The pictures show the graphs of some functions. In each case, tell whether the left and right-hand limits at c are defined. If both are defined, tell whether they are equal.



(a) (b) (c)

In (a), the right-hand limit is defined, because the graph approaches a definite height from the right (the height of the dot). The left-hand limit is undefined because the graph is not approaching a definite height: There is a vertical asymptote. (You could also say the left-hand limit is $+\infty$, as we'll discuss below.)

Likewise, in (b), the right-hand limit is undefined, and the left-hand limit is defined. (You could also say the right-hand limit is $+\infty$, as we'll discuss below.)

Finally, in (c), both the right and left-hand limits are defined, but they aren't equal. (This means that the ordinary ("two-sided") limit $\displaystyle\lim_{x\to c} f(x)$ is undefined. □

I won't state a lot of theorems about left and right-hand limits, because in general the results that hold for ordinary ("two-sided") limits hold for one-sided limits. For example (omitting the usual technical assumptions), here is the rule for sums for right-hand limits:

$$\lim_{x\to c^+}(f(x)+g(x)) = \lim_{x\to c^+} f(x) + \lim_{x\to c^+} g(x).$$

You can see that it's the same as the rule for sums for ordinary limits, the only difference being that I'm now writing "$x \to c^+$" instead of "$x \to c$".

One important point which we've already noted is the relationship between left and right-hand limits and ordinary ("two-sided") limits. To give a little more detail, I'll first give the formal definitions for left and right-hand limits.

Definition. (a) (**Right-hand limits**) Suppose $f(x)$ is defined on an interval (c, b) for $c < b$. To say that $\lim_{x\to c^+} f(x) = L$ means: For every number $\epsilon > 0$, there is a number δ, such that

$$\text{if} \quad c < x < c + \delta, \quad \text{then} \quad \epsilon > |f(x) - L|.$$

(b) (**Left-hand limits**) Suppose $f(x)$ is defined on an interval (a, c) for $c < b$. To say that $\lim_{x\to c^-} f(x) = L$ means: For every number $\epsilon > 0$, there is a number δ, such that

$$\text{if} \quad c - \delta < x < c, \quad \text{then} \quad \epsilon > |f(x) - L|.$$

Note that in each case, $f(x)$ might actually be defined on both sides of c. We're saying that for the right-hand limit to exist, it *only needs to be defined* to the right of c; for the left-hand limit to exist, it *only needs to be defined* to the left of c. (As usual, $f(x)$ may or may not be defined at c.)

Here's the result which we've used informally before that relates left and right-hand limits to ordinary ("two-sided") limits. The proof is an ϵ-δ proof like the ones I gave in the sections on the definition of limits and limit theorems; if you're in an ordinary first-term calculus course, you can skip the proof if you wish.

Theorem. Suppose $f(x)$ is defined on an open interval containing c.

Then $\lim_{x\to c} f(x)$ is defined if and only if $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c-} f(x)$ are both defined and equal.

In this case, $\lim_{x\to c} f(x)$ is equal to the common value of $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c-} f(x)$.

$$\lim_{x\to c} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c-} f(x).$$

Proof. The proof of this theorem comes down to the following fact about absolute values:

$$\delta > |x - c| > 0 \quad \text{if and only if either} \quad c - \delta < x < c \quad \text{or} \quad c < x < c + \delta.$$

The reason is that $\delta > |x - c| > 0$ means that x is within δ of c, but is not equal to c.

On the other hand, $c - \delta < x < c$ means that x is less than c and is within δ of c, and $c < x < c + \delta$ means that x is greater than c and is within δ of c. Thus, if one of these two statements is true, then the previous statement is true, and if the previous statement is true, then one of these must be true.

Thus, suppose $\lim_{x\to c} f(x) = L$. I'll show that

$$\lim_{x\to c^+} f(x) = L \quad \text{and} \quad \lim_{x\to c-} f(x) = L.$$

Let $\epsilon > 0$. Since $\lim_{x\to c} f(x) = L$, there is a number δ such that if $\delta > |x - c| > 0$, then $\epsilon > |f(x) - L|$.

First, if $c < x < c + \delta$, then $\delta > |x - c| > 0$. Consequently, $\lim_{x\to c^+} f(x) = L$.

Second, if $c - \delta < x < c$, then $\delta > |x - c| > 0$. Consequently, $\lim_{x\to c^-} f(x) = L$.

Next, I'll prove that converse. Suppose that

$$\lim_{x\to c^+} f(x) = L \quad \text{and} \quad \lim_{x\to c-} f(x) = L.$$

I'll show that

$$\lim_{x\to c} f(x) = L.$$

Let $\epsilon > 0$. Since $\lim_{x\to c^+} f(x) = L$, there's a number δ_1 such that if $c < x < c + \delta_1$, then $\epsilon > |f(x) - L|$.

Likewise, since $\lim_{x\to c^-} f(x) = L$, there's a number δ_2 such that if $c - \delta_2 < x < c$, then $\epsilon > |f(x) - L|$.

Now let $\delta = \min(\delta_1, \delta_2)$. Remember that this means that δ is the smaller of δ_1 and δ_2, so it's at least as small as either.

Suppose $\delta > |x - c| > 0$. This means that either

$$c - \delta < x < c \quad \text{or} \quad c < x < c + \delta.$$

In the first case, I have

$$c - \delta_2 \le c - \delta < x < c.$$

Hence, $\epsilon > |f(x) - L|$.
In the second case, I have

$$c < x < x + \delta \le x + \delta_1.$$

Hence, $\epsilon > |f(x) - L|$.
This proves that $\lim_{x \to c} f(x) = L$. □

In words, this result says that the ordinary ("two-sided") limit is defined if and only if the left and right-hand limits are defined and equal, and in that case, their common value is the value of the ordinary limit.

Example. Compute

$$\lim_{x \to 0+} \frac{|\sin x|}{\sin x} \quad \text{and} \quad \lim_{x \to 0-} \frac{|\sin x|}{\sin x}.$$

Is $\lim_{x \to 0} \dfrac{|\sin x|}{\sin x}$ defined?

$$\lim_{x \to 0+} \frac{|\sin x|}{\sin x} = 1, \quad \text{but} \quad \lim_{x \to 0-} \frac{|\sin x|}{\sin x} = -1.$$

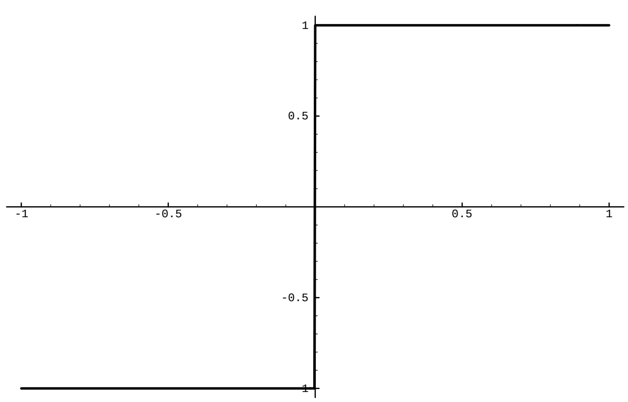
Look at the first limit more closely. x approaches 0 from the *right*. Numbers close to, but to the right of, 0 are small positive numbers: 0.01, for example. Small positive numbers make $\sin x$ positive: $\sin 0.01 \approx 0.01000$, for example. If $\sin x$ is positive, then $|\sin x| = \sin x$, so

$$\frac{|\sin x|}{\sin x} = \frac{\sin x}{\sin x} = 1.$$

(Notice that you don't let x *equal* 0, so $\sin x \neq 0$, and the cancellation is legal.)
Therefore,

$$\lim_{x \to 0+} \frac{|\sin x|}{\sin x} = \lim_{x \to 0+} 1 = 1.$$

Here's the picture:



Since the left- and right-hand limits are not the same, $\lim_{x \to 0} \dfrac{|\sin x|}{\sin x}$ is undefined. □

Example. Suppose

$$f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 5 & \text{if } x = 1 \\ 7x^2 - 4 & \text{if } x > 1 \end{cases}.$$

Compute $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^-} f(x)$, and $\lim_{x\to 1} f(x)$.

To compute $\lim_{x\to 1^+} f(x)$, I use the part of the definition for f which applies to $x > 1$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x+1) = 3.$$

Likewise, to compute $\lim_{x\to 1^-} f(x)$, I use the part of the definition for f which applies to $x < 1$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (7x^2 - 4) = 3.$$

Since the left and right-hand limits are equal, the two-sided limit is defined, and $\lim_{x\to 1} f(x) = 3$. The fact that $f(1) = 5$ does not come into the problem. $\square$

Example. A function is defined by

$$f(x) = \begin{cases} kx+7 & \text{if } x \ge 2 \\ x^2 + 19 & \text{if } x < 2 \end{cases}.$$

For what value of k is $\lim_{x\to 2} f(x)$ defined?

In order for $\lim_{x\to 2} f(x)$ to be defined, the left and right-hand limits at 2 must be defined and equal. Compute them:

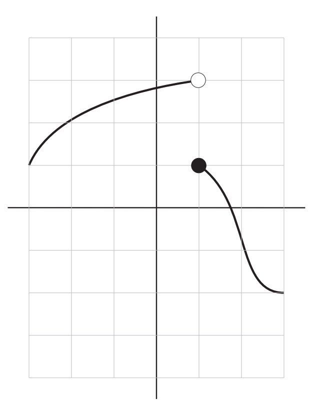
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (kx+7) = 2k+7.$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2+19) = 23.$$

Set the left and right-hand limits equal and solve for k:

$$\begin{aligned} 2k + 7 &= 23 \\ 2k &= 16 \quad \square \\ k &= 8 \end{aligned}$$

Example. Consider the function $f(x)$ whose graph is depicted below:



Compute $\lim_{x \to 1} f(x)$.

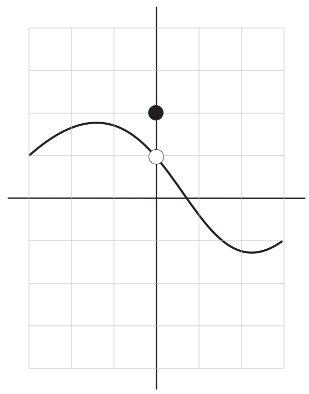
Then

$$\lim_{x \to 1^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 1^-} f(x) = 3.$$

Since the left- and right-hand limits are not the same,

$$\lim_{x \to 1} f(x) \quad \text{is undefined.} \quad \square$$

Example. Consider the function $f(x)$ whose graph is depicted below:



Compute

$$\lim_{x \to 0^+} f(x) \quad \text{and} \quad \lim_{x \to 0^-} f(x).$$

Do these limits depend on the value of $f(0)$?

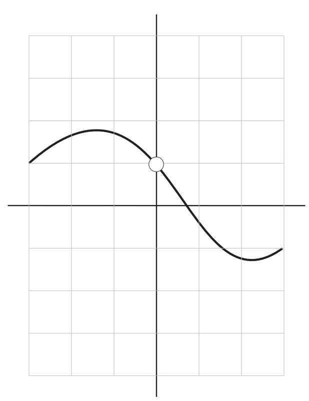
Then

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = 1.$$

Therefore,

$$\lim_{x \to 0} f(x) = 1.$$

The value of $f(0)$ does not affect the existence of the limit. In fact, suppose I change the function as follows:



Now $f(0)$ is undefined, but

$$\lim_{x \to 0^+} f(x) = 1, \quad \lim_{x \to 0^-} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0} f(x) = 1. \quad \square$$

Left and right-hand limits can give rise to **infinite limits**, so I'll discuss the ideas briefly before giving some examples. As usual with the theory in this course, the precise definitions are here for the sake of completeness, and for people who are interested. For most people, it's enough that you have a good grasp of *how it looks graphically* when a limit is infinite, and how infinite limits can arise in limit computations.

Definition. (a) $\lim_{x\to c} f(x) = \infty$ means: For every number $M > 0$, there is a number δ, such that if $\delta > |x - c| > 0$, then $f(x) > M$.

Sometimes I'll write "$+\infty$" instead of "∞" for emphasis, to help distinguish it from "$-\infty$" in the next part of the definition.

The definitions for right and left-hand limits are:

(i) (Right-hand limits) $\lim_{x\to c^+} f(x) = \infty$ means: For every number $M > 0$, there is a number δ, such that if $c < x < c + \delta$, then $f(x) > M$.

(ii) (Left-hand limits) $\lim_{x\to c^-} f(x) = \infty$ means: For every number $M > 0$, there is a number δ, such that if $c - \delta < x < c$, then $f(x) > M$.

(b) $\lim_{x\to c} f(x) = -\infty$ means: For every number $M < 0$, there is a number δ, such that if $\delta > |x - c| > 0$, then $f(x) < M$.

The definitions for right and left-hand limits are:

(i) (Right-hand limits) $\lim_{x\to c^+} f(x) = -\infty$ means: For every number $M < 0$, there is a number δ, such that if $c < x < c + \delta$, then $f(x) < M$.

(ii) (Left-hand limits) $\lim_{x\to c^-} f(x) = -\infty$ means: For every number $M < 0$, there is a number δ, such that if $c - \delta < x < c$, then $f(x) < M$.

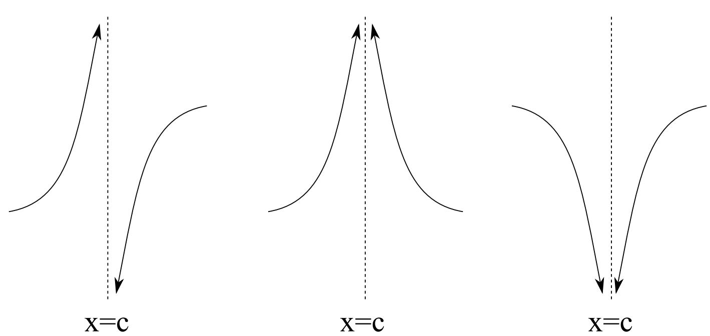
Thus, to say $f(x)$ approaches $+\infty$ as x approaches c (from the left, the right, or from both sides) means that as $f(x)$ becomes larger and positive, without any upper bound, as x approaches c.

Likewise, to say $f(x)$ approaches $-\infty$ as x approaches c (from the left, the right, or from both sides) means that as $f(x)$ becomes larger and negative, without any upper bound, as x approaches c.

In all of these cases, it would not be wrong to say that the limit is undefined, in the sense that it is not a *number*. But if you can say it is $+\infty$ or $-\infty$, it is better, since you're giving more information about what is happening.

Example. Each picture below shows the graph of a function $f(x)$. In each case, find:

$$\lim_{x\to c^+} f(x), \quad \lim_{x\to c^-} f(x), \quad \text{and} \quad \lim_{x\to c} f(x).$$



(a) (b) (c)

In (a),

$$\lim_{x\to c^+} f(x) = +\infty \quad \text{and} \quad \lim_{x\to c^-} f(x) = -\infty.$$

Since the left and right-hand limits do not agree, $\lim_{x\to c} f(x)$ is undefined.

In (b),

$$\lim_{x\to c^+} f(x) = +\infty, \quad \lim_{x\to c^-} f(x) = +\infty, \quad \lim_{x\to c} f(x) = +\infty.$$

$$\lim_{x\to c^+} f(x) = -\infty, \quad \lim_{x\to c^-} f(x) = -\infty, \quad \lim_{x\to c} f(x) = -\infty. \quad \square$$

Example. Compute $\lim_{x\to 1+} \dfrac{x^2 - 2x - 3}{x - 1}$.

Plugging in gives $\dfrac{-4}{0}$. The limit is *undefined.* But I can say more.

Try plugging in a number close to 1: When $x = 1.001$,

$$\frac{x^2 - 2x - 3}{x - 1} \approx -4000.$$

It looks as though $\dfrac{x^2 - 2x - 3}{x - 1}$ is getting *big and negative.* In fact,

$$\lim_{x\to 1+} \frac{x^2 - 2x - 3}{x - 1} = -\infty.$$

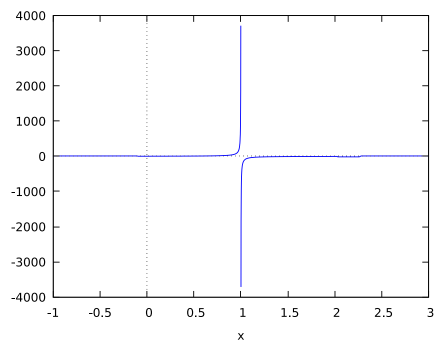
To why this is true, remember that x is approaching 1 *from the right.* This means that $x - 1$ will be small and positive. On the other hand, $x^2 - 2x - 3 \to -4$. Since the top is negative and the bottom is positive, the result must be *negative.*

As far as size goes, I have

$$\frac{\text{nonzero number}}{\text{small number}} = \text{big number}.$$

Since the result should be *big* and *negative*, it is reasonable that it is $-\infty$.

Another way to see this is to draw the graph near $x = 1$. As you move toward 1 from the right, the graph goes downward toward $-\infty$.



$\square$

I noted the following fact earlier: Suppose

$$\frac{f(x)}{g(x)} \to \frac{\text{nonzero number}}{0}.$$

Then the two-sided limit $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ is *undefined.* As the example above shows, the situation is different with one-sided limits.

If in this situation $g(x)$ *has the same sign* for all x's sufficiently close to c and greater than c, then the right-hand limit $\lim_{x \to c^+} \dfrac{f(x)}{g(x)}$ will be either $+\infty$ or $-\infty$. The specific sign depends on the signs of the top and the bottom of the fraction.

Likewise, if $g(x)$ *has the same sign* for all x's sufficiently close to c and less than c, then the left-hand limit $\lim_{x \to c^-} \dfrac{f(x)}{g(x)}$ will be either $+\infty$ or $-\infty$. Again, the specific sign depends on the signs of the top and the bottom of the fraction.

The "same-sign" condition will be satisfied, for example, if f and g are polynomials — that is, if $\dfrac{f(x)}{g(x)}$ is a rational function. It will also be satisfied by functions like

$$\frac{x-3}{x^{1/3}-2} \quad \text{as} \quad x \to 8.$$

Example. Compute $\lim_{x \to -3^+} \dfrac{x+1}{x+3}$.

Plugging $x = -3$ in gives $\dfrac{-2}{0}$. Since $\dfrac{x+1}{x+3}$ is a rational function, the right-hand limit $\lim_{x \to -3^+} \dfrac{x+1}{x+3}$ is either $+\infty$ or $-\infty$; I have to determine which of the two it is. I'll look at the top and the bottom separately.

As $x \to -3^+$, $x + 1 \to -2$.

As for the bottom, since x is approaching -3 from the *right*, I'm considering x's *greater than* -3. Thus, $x > -3$, so $x + 3 > 0$ — $x + 3$ is positive.

Since $x + 1$ is approaching a negative number and $x + 3$ is approaching a positive number, the quotient is negative. Therefore,

$$\lim_{x \to -3^+} \frac{x+1}{x+3} = -\infty.$$

I can also see this if I take a number close to -3 but to the right of -3 — $x = -2.99$, for example — and plug it in:

$$\frac{x+1}{x+3} = \frac{-2.99+1}{-2.99+3} = \frac{-1.99}{0.01} = -199.$$

I got a large *negative* number, which suggests that the limit should be $-\infty$.

I could also see this by graphing the function, as in the previous example.

In the case of a one-sided limit and a form $\dfrac{\text{nonzero number}}{0}$, you might ask: "Which of these methods is *the best* to determine the value?" I feel that for a first course in calculus, all three are *acceptable.*

However, while plugging in numbers and drawing graphs provide *support* for a conclusion, they don't really provide a *proof.* Graphs can be deceiving. And when you plug in a number, how do you know that the number you chose is "typical"? The first method — reasoning about signs using inequalities — is much closer to a rigorous proof of the result.