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L'Hôpital's Rule

L'Hôpital's Rule is a method for computing a limit of the form

$$\lim_{x \to c} \frac{f(x)}{g(x)}.$$

c can be a number, $+\infty$, or $-\infty$. The conditions for applying it are:

1. The functions f and g are differentiable in an open interval containing c. (c may also be an endpoint of the open interval, if the limit is one-sided.)

2. g and g' are nonzero in the open interval, except possibly at c.

3. $\displaystyle\lim_{x \to c} \frac{f'(x)}{g'(x)}$ is defined, or is $+\infty$, or is $-\infty$.

4. As $x \to c$,

$$\frac{f(x)}{g(x)} \to \frac{0}{0} \qquad \text{or} \qquad \frac{\infty}{\infty}.$$

If these conditions hold, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

In other words, f and g may be replaced by their derivatives.

Note that you're *not* applying the Quotient Rule to $\displaystyle\frac{f(x)}{g(x)}$.

A proof may be given using some advanced results (e.g. the Extended Mean Value Theorem). I won't give a proof here.

Example. Compute

$$\lim_{x \to 1} \frac{4x^{100} - x - 3}{5x^{50} - x - 4}.$$

Plugging $x = 1$ into $\displaystyle\frac{4x^{100} - x - 3}{5x^{50} - x - 4}$ gives $\displaystyle\frac{0}{0}$, so I can apply L'Hôpital's Rule:

$$\lim_{x \to 1} \frac{4x^{100} - x - 3}{5x^{50} - x - 4} = \lim_{x \to 1} \frac{400x^{99} - 1}{250x^{49} - 1} = \frac{400 - 1}{250 - 1} = \frac{399}{249}. \quad \square$$

Example. Compute

$$\lim_{x \to 2} \frac{\sin(x^2 - 4)}{x - 2}.$$

Plugging $x = 2$ into $\displaystyle\frac{\sin(x^2 - 4)}{x - 2}$ gives $\displaystyle\frac{0}{0}$, so I can apply L'Hôpital's Rule:

$$\lim_{x \to 2} \frac{\sin(x^2 - 4)}{x - 2} = \lim_{x \to 2} \frac{2x\cos(x^2 - 4)}{1} = 4. \quad \square$$

Example. Compute

$$\lim_{x\to+\infty}\frac{\ln(x-4)}{x}.$$

As $x\to+\infty$, $\dfrac{\ln(x-4)}{x}\to\dfrac{\infty}{\infty}$, so I can apply L'Hôpital's Rule:

$$\lim_{x\to+\infty}\frac{\ln(x-4)}{x}=\lim_{x\to+\infty}\frac{\frac{1}{x-4}}{1}=0.\quad\square$$

Example. Compute

$$\lim_{x\to 0^+}\frac{x^2+3x+7}{x}.$$

As $x\to 0^+$, $\dfrac{x^2+3x+7}{x}\to\dfrac{7}{0}$, so I *can't* apply L'Hôpital's Rule. In fact, since the top and bottom are both positive,

$$\lim_{x\to 0^+}\frac{x^2+3x+7}{x}=+\infty.\quad\square$$

Example. Compute

$$\lim_{x\to\infty}\left(\sqrt{x^2+6x}-x\right).$$

As $x\to+\infty$, $\sqrt{x^2+6x}-x\to\infty-\infty$ (which is *not* 0!). I convert the expression into a fraction by **rationalizing**:

$$\lim_{x\to\infty}\left(\sqrt{x^2+6x}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+6x}-x\right)\cdot\frac{\sqrt{x^2+6x}+x}{\sqrt{x^2+6x}+x}=\lim_{x\to\infty}\frac{x^2+6x-x^2}{\sqrt{x^2+6x}+x}=\lim_{x\to\infty}\frac{6x}{\sqrt{x^2+6x}+x}.$$

As $x\to+\infty$, $\dfrac{6x}{\sqrt{x^2+6x}+x}\to\dfrac{\infty}{\infty}$, so I could apply L'Hôpital's Rule. Instead, I'll divide the top and bottom by x:

$$\lim_{x\to\infty}\frac{6x}{\sqrt{x^2+6x}+x}=\lim_{x\to\infty}\frac{6}{\sqrt{1+\frac{6}{x}}+1}=\frac{6}{1+1}=3.\quad\square$$

If you apply L'Hôpital's Rule, and the limit you obtain is undefined, you may not conclude that the original limit is undefined.

Example. Compute

$$\lim_{x\to\infty}\frac{x-\sin x}{x+\sin x}.$$

As $x\to+\infty$, $\dfrac{x-\sin x}{x+\sin x}\to\infty\infty$, so I can apply L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{x-\sin x}{x+\sin x}=\lim_{x\to\infty}\frac{1-\cos x}{1+\cos x}=\lim_{x\to\infty}\frac{\frac{1}{2}(1-\cos x)}{\frac{1}{2}(1+\cos x)}=\lim_{x\to\infty}\frac{\left(\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}\right)^2}=\lim_{x\to\infty}\left(\tan\frac{x}{2}\right)^2.$$

The last limit is undefined, because $\tan \dfrac{x}{2}$ has no limit as $x \to \infty$. This implies that the ='s in the reasoning above aren't valid. When you do a L'Hôpital computation, the equalities are actually provisional, pending the existence of a limit in the chain.

In fact, the original limit exists:

$$\lim_{x\to\infty} \frac{x - \sin x}{x + \sin x} = \lim_{x\to\infty} \frac{1 - \dfrac{\sin x}{x}}{1 + \dfrac{\sin x}{x}} = \frac{1-0}{1+0} = 1. \quad \square$$

You can handle the indeterminate form $0 \cdot \infty$ by using algebra to convert the expression to a fraction, and then applying L'Hôpital's Rule.

Example. Compute

$$\lim_{x\to\pi/2^-} \left(x - \frac{\pi}{2}\right) \tan x.$$

As $x \to \pi/2^-$, $\left(x - \dfrac{\pi}{2}\right) \tan x \to 0 \cdot \infty$. So

$$\lim_{x\to\pi/2^-} \left(x - \frac{\pi}{2}\right) \tan x = \lim_{x\to\pi/2^-} \frac{x - \dfrac{\pi}{2}}{\cot x}.$$

As $x \to \pi/2^-$, $\dfrac{x - \dfrac{\pi}{2}}{\cot x} \to \dfrac{0}{0}$, so I can apply L'Hôpital's Rule:

$$\lim_{x\to\pi/2^-} \frac{x - \dfrac{\pi}{2}}{\cot x} = \lim_{x\to\pi/2^-} \frac{1}{-(\csc x)^2} = \lim_{x\to\pi/2^-} -(\sin x)^2 = -1. \quad \square$$

The indeterminate form 1^∞ can be handled by taking logs, computing the limit using the techniques above, and finally exponentiating to undo the log.

Example. Compute

$$\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{5x}.$$

As $x \to \infty$, $\left(1 + \dfrac{3}{x}\right)^{5x} \to 1^\infty$.

Let $y = \left(1 + \dfrac{3}{x}\right)^{5x}$. Then

$$\ln y = \ln \left(1 + \frac{3}{x}\right)^{5x} = 5x \ln \left(1 + \frac{3}{x}\right).$$

So

$$\begin{aligned}
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} 5x \ln\left(1+\frac{3}{x}\right) \\
&= 5\lim_{x\to\infty} \frac{\ln\left(1+\dfrac{3}{x}\right)}{\dfrac{1}{x}} \\
&= 5\lim_{x\to\infty} \frac{\dfrac{1}{1+\dfrac{3}{x}}\left(-\dfrac{3}{x^2}\right)}{\dfrac{-1}{x^2}} \\
&= 5\lim_{x\to\infty} \frac{3}{1+\dfrac{3}{x}} \\
&= 15
\end{aligned}$$

Therefore,

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{5x} = e^{15}. \quad \square$$

Example. Compute

$$\lim_{x\to\infty}\left(\frac{x+1}{x-2}\right)^{2x-1}.$$

As $x\to\infty$, $\left(\dfrac{x+1}{x-2}\right)^{2x-1} \to 1^{\infty}$.

Let $y = \left(\dfrac{x+1}{x-2}\right)^{2x-1}$. Then

$$\ln y = \ln\left(\frac{x+1}{x-2}\right)^{2x-1} = (2x-1)\ln\left(\frac{x+1}{x-2}\right).$$

So

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} (2x-1)\ln\left(\frac{x+1}{x-2}\right).$$

As $x\to\infty$, $(2x-1)\ln\left(\dfrac{x+1}{x-2}\right) \to \infty\cdot 0$. So convert the expression to a fraction:

$$\lim_{x\to\infty} (2x-1)\ln\left(\frac{x+1}{x-2}\right) = \lim_{x\to\infty} \frac{\ln\left(\dfrac{x+1}{x-2}\right)}{\dfrac{1}{2x-1}}.$$

As $x\to\infty$, $\dfrac{\ln\left(\dfrac{x+1}{x-2}\right)}{\dfrac{1}{2x-1}} \to \dfrac{0}{0}$, so I can apply L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln\left(\dfrac{x+1}{x-2}\right)}{\dfrac{1}{2x-1}} = \lim_{x\to\infty} \frac{\left(\dfrac{x-2}{x+1}\right)\left(\dfrac{(x-2)(1)-(x+1)(1)}{(x-2)^2}\right)}{\dfrac{-2}{(2x-1)^2}} = \lim_{x\to\infty} \frac{\dfrac{-3}{(x+1)(x-2)}}{\dfrac{-2}{(2x-1)^2}} =$$

$$\frac{3}{2}\lim_{x\to\infty}\frac{(2x-1)^2}{(x+1)(x-2)} = 6.$$

That is, $\lim_{x\to\infty}\ln y = 6$. Therefore,

$$\lim_{x\to\infty} y = e^{(\lim_{x\to\infty}\ln y)} = e^6 = 403.42879\ldots. \quad \square$$

Example. Compute $\displaystyle\lim_{x\to 1^+}\left(\tan\frac{\pi x}{4}\right)^{3/(x-1)}$.

As $x \to 1^+$, $\left(\tan\frac{\pi x}{4}\right)^{3/(x-1)} \to 1^\infty$. Set $y = \left(\tan\frac{\pi x}{4}\right)^{3/(x-1)}$. Take logs and simplify:

$$\ln y = \ln\left(\tan\frac{\pi x}{4}\right)^{3/(x-1)} = 3\cdot\frac{\ln\left(\tan\frac{\pi x}{4}\right)}{x-1}.$$

Take the limit as $x \to 1^+$, applying L'Hôpital's rule to the fraction:

$$\lim_{x\to 1^+}\ln y = \lim_{x\to 1^+} 3\cdot\frac{\ln\left(\tan\frac{\pi x}{4}\right)}{x-1} = 3\lim_{x\to 1^+}\frac{\left(\dfrac{1}{\tan\frac{\pi x}{4}}\right)\left(\sec\frac{\pi x}{4}\right)^2\left(\frac{\pi}{4}\right)}{1} = \frac{3\pi}{2}.$$

Hence, $\displaystyle\lim_{x\to 1^+}\left(\tan\frac{\pi x}{4}\right)^{3/(x-1)} = e^{3\pi/2}$. $\square$

Example. Compute $\displaystyle\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$.

This is an indeterminate form $\frac{1}{0} - \frac{1}{0}$. Combine the fractions over a common denominator:

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\frac{x\ln x - (x-1)}{(x-1)\ln x} = \lim_{x\to 1}\frac{x\ln x - x + 1}{(x-1)\ln x}.$$

This is an $\frac{0}{0}$ form, so I can apply L'Hôpital's Rule:

$$\lim_{x\to 1}\frac{x\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1}\frac{1 + \ln x - 1}{\dfrac{x-1}{x} + \ln x} = \lim_{x\to 1}\frac{\ln x}{1 - \dfrac{1}{x} + \ln x} = \lim_{x\to 1}\frac{\dfrac{1}{x}}{\dfrac{1}{x^2} + \dfrac{1}{x}} = \frac{1}{2}. \quad \square$$
