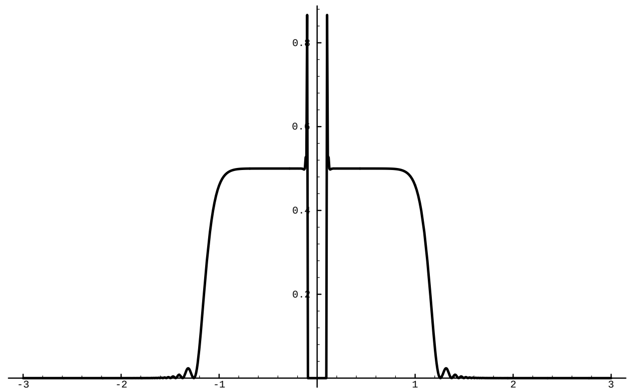
8-9-2018

The Limit Definition

Having discussed how you can compute limits, I want to examine the definition of a limit in more detail.

You might wonder why it is necessary to be careful. Suppose you're trying to compute $\lim_{x \to 0} \dfrac{1 - \cos(x^8)}{x^{16}}$. You might think of drawing a graph; many graphing calculators, for instance, produce a graph like the one below:



It looks as though the graph is dropping down to 0 near $x = 0$. From this, you might guess that the limit is 0. In fact,

$$\lim_{x \to 0} \frac{1 - \cos(x^8)}{x^{16}} = \frac{1}{2}.$$

It's possible to justify this algebraically once you know a little about limits of trig functions.

Pictures can be helpful; so can experimenting with numbers. In many cases, pictures and numerical experiments are inconclusive or even misleading. In these cases, how can you determine whether a proposed answer is correct or not?

Because the limit definition is a bit abstract, I'll start off with an informal definition.

Informal Definition. If $f(x)$ can be made arbitrarily close to L for all x's sufficiently close to c, then

$$\lim_{x \to c} f(x) = L.$$

This statement is like a guarantee. Think of making parts in a factory. Your customers won't buy your parts unless they meet certain specifications. So you might guarantee that your parts will be within 0.01 of the customer's specification.

Likewise, to say that $\lim_{x \to c} f(x) = L$ you must be able to guarantee that you can make $f(x)$ fall within (say) 0.01 of L. But you have to do more: You have to be able to make $f(x)$ fall within *any positive tolerance* of L — 0.0001, 0.0000004, and so on, no matter how small.

Another way to think of this is as *meeting a challenge*; for example:

Challenge: "I challenge you to make $f(x)$ stay within 0.0005 of L."

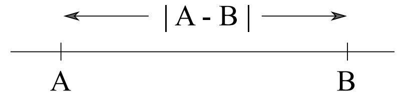
Your response: "I guarantee that every x within 0.003 of c (except perhaps c itself) will give an $f(x)$ that is within 0.0005 of L."

To prove that $\lim_{x \to c} f(x) = L$, you must be able to meet the challenge *no matter what positive number* is used in place of 0.0005.

By the way, notice that $x = c$ is excluded in my guarantee. The reason is that in computing $\lim_{x \to c} f(x)$, we're concerned with what happens as x *approaches* c, not what $f(c)$ is.

Before I give some examples, here's an important fact about absolute value:

$$|A - B| = (\text{the distance from A to B}).$$



We want absolute values, which are always *nonnegative*, because a distance shouldn't be negative. Also, notice that

$$|A - B| = |B - A|.$$

That makes sense, because the distance from A to B should be the same as the distance from B to A. For instance,

$$|8 - 2| = |6| = 6 \quad \text{and} \quad |2 - 8| = |-6| = 6.$$

Example. By plugging in $x = 4$, it appears that

$$\lim_{x \to 4} (3x - 5) = 7.$$

How close should x be to 4 to guarantee that $3x - 5$ is within 0.01 of 7?

Let's work backwards: I want $3x - 5$ to be within 0.01 of 7. This means

$$\begin{aligned} |(3x - 5) - 7| &< 0.01 \\ |3x - 12| &< 0.01 \\ 3|x - 4| &< 0.01 \\ |x - 4| &< \frac{0.01}{3} \end{aligned}$$

The last inequality says that the distance from x to 4 should be less than $\frac{0.01}{3}$. So if x lies within $\frac{0.01}{3}$ of 4, I can guarantee that $3x - 5$ will be within 0.01 of 7.

A formal proof would just reverse the steps above:

$$\begin{aligned} |x - 4| &< \frac{0.01}{3} \\ 3|x - 4| &< 0.01 \\ |3x - 12| &< 0.01 \\ |(3x - 5) - 7| &< 0.01 \end{aligned}$$

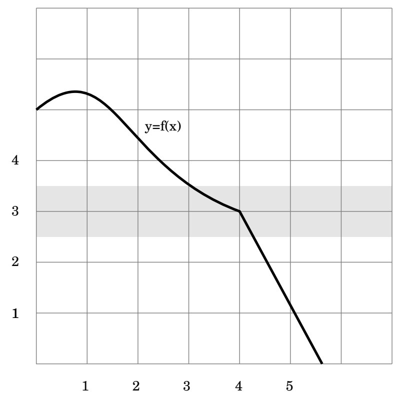
Can you see that if I'm challenged to make $3x - 5$ lie within 0.00001 of 7, I should make x lie within $\frac{0.00001}{3}$ of 4? Just replace 0.01 with 0.00001 in the discussion above.

And similarly, I can make $3x - 5$ lie within any tolerance FOO of 7 by making x lie within $\frac{\text{FOO}}{3}$ of 4.

This shows that I can meet *any challenge*, since I can just take the challenge tolerance and plug it in for FOO. This proves that

$$\lim_{x \to 4} (3x - 5) = 7. \quad \square$$

Example. The graph of a function $y = f(x)$ is shown below.



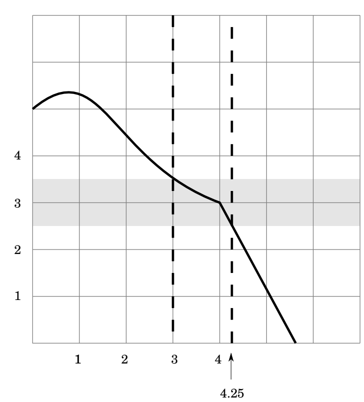
It appears that $\lim_{x \to 4} f(x) = 3$.

A (grey) horizontal strip of width 0.5 is drawn around $y = 3$. Draw a picture to show a range of x-values around 4 for which the corresponding $f(x)$-values lie in the horizontal strip.

Use it to estimate the width of a symmetric vertical strip around 4 representing x-values whose corresponding $f(x)$-values lie in the horizontal strip.

Suppose I'm challenged to make $f(x)$ fall within 0.5 of 3. That is, I want my y-values to fall within the grey strip in the picture.

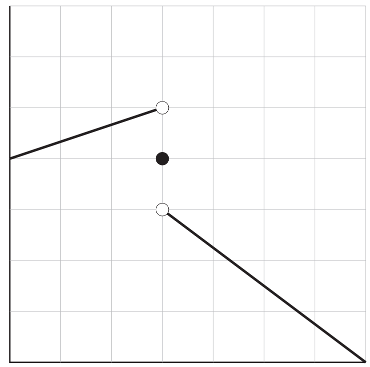
On the right side of 4, the graph stays within the grey strip as far as 4.25; on the left side of 4, the graph stays within the grey strip as far as 3.



If I want a strip that's symmetric about 4, I use the closer of the two values, which is 4.25. Now 4.25 is 0.25 units from 4, so my answer is: If x is within 0.25 of 4, then $f(x)$ will be within 0.5 of 3. □

If I can meet such a challenge with *any positive number* in place of 0.5, then I will have proved that $\lim_{x \to 4} f(x) = 3$.

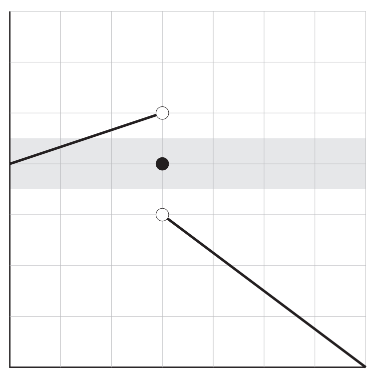
Example. (Disproving a limit) Consider the function $y = f(x)$ whose graph is show below.



Suppose that Calvin Butterball thinks that $\lim_{x\to 3} f(x) = 4$. Use the limit definition to disprove it.

To disprove Calvin's claim, I'll make a challenge that Calvin can't meet.

I challenge Calvin to make $f(x)$ fall within 0.5 of 4. This means that he must find a range of x's around 3 so that the corresponding part of the graph lies within the grey strip shown below:



You can see that there's no way to do this. (Note: He's not allowed to use $x = 3$ alone. Remember that what the function does *at* $x = 3$ has no bearing on the value of the limit.)

Since this challenge can't be met, $\lim_{x\to 3} f(x) \neq 4$. In fact, $\lim_{x\to 3} f(x)$ is *undefined*. □

Example. Suppose

$$f(x) = \begin{cases} 5 - 2x & \text{if } x < 1 \\ 4x - 1 & \text{if } x \geq 1 \end{cases}.$$

It appears that $\lim_{x\to 1} f(x) = 3$. How close should x be to 1 in order to guarantee that $f(x)$ will be within 0.0008 of 3?

As in an earlier example, I'll work backwards.

From the left side, I'd need

$$\begin{aligned} |(5 - 2x) - 3| &< 0.0008 \\ |2 - 2x| &< 0.0008 \\ |2x - 2| &< 0.0008 \\ |x - 1| &< 0.0004 \end{aligned}$$

The last inequality says that x should be within 0.0004 of 1.

From the right side, I'd need

$$\begin{aligned} |(4x-1)-3| &< 0.0008 \\ |4x-4| &< 0.0008 \\ |x-1| &< 0.0002 \end{aligned}$$

This means that x should be within 0.0002 of 1.

To satisfy the two requirements at the same time, I'll use the smaller of the two numbers. So I'll require that x should be within 0.0002 of 1, which means

$$|x-1| < 0.0002.$$

Here is the "real" proof, which I get by writing the scratch work in the reverse order.

Suppose $|x-1| < 0.0002$. If $x \ge 1$, I have

$$\begin{aligned} |x-1| &< 0.0002 \\ |4x-4| &< 0.0008 \\ |(4x-1)-3| &< 0.0008 \\ |f(x)-3| &< 0.0008 \end{aligned}$$

Now

$$|x-1| < 0.0002 < 0.0004.$$

So if $x < 1$, I have

$$\begin{aligned} |x-1| &< 0.0004 \\ |2x-2| &< 0.0008 \\ |2-2x| &< 0.0008 \\ |(5-2x)-3| &< 0.0008 \\ |f(x)-3| &< 0.0008 \end{aligned}$$

(From the second to the third line, I used the fact that $|A-B| = |B-A|$.)

Thus, if x is within 0.0002 of 1, then $f(x)$ will be within 0.0008 of 3. □

I'm almost ready to give the formal definition of a limit, but I need to mention something first as a matter of honesty. It's a technical issue, and it won't arise in the majority of problems and examples (so you can ignore it without much harm if you wish).

A technical point. In discussing $\lim_{x \to c} f(x)$, I'll usually assume that f is defined on an open interval containing c. That is, there are numbers a and b such that $a < c < b$ and f is defined (at least) on $a < x < b$.

For one-sided limit (which I'll discuss later), $f(x)$ should be defined on an open interval with c as an endpoint.

To understand why you want to do this, consider the function

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ 42 & \text{if } x = -10 \end{cases}.$$

(So, for instance, f is simply not defined at $x = -1$, or at $x = -57$.)

In the definition of $\lim_{x \to -10} f(x)$, the "if" part of the definition would hold vacuously (for small open intervals around -10), because there would be *no values* of x near -10 for which f was defined. Thus, the limit L could be anything!

The condition on the domain of f is made to avoid silly cases like this one.

In order to avoid cluttering the statements of the definition or of proofs of limit properties, I usually won't state this assumption about the domains of functions in limits explicitly.

Now I'll give the formal definition of a limit, and show how to use it to do ϵ-δ proofs.

Definition. $\displaystyle\lim_{x\to c} f(x) = L$ means:

For every $\epsilon > 0$, there is a δ, such that for all x in the domain of f, if $\delta > |x-c| > 0$, then $\epsilon > |f(x)-L|$.

"ϵ" is the Greek letter epsilon. It is the "challenge number", the tolerance or maximum error you have to meet. δ is the Greek letter delta. It is the "response number", the setting on x which meets the challenge. The Greek letters are used in this definition for traditional reaons; there is nothing otherwise special about them.

Let's see how proofs of limits work using the definition.

Example. Prove that $\displaystyle\lim_{x\to 4}(7x-3) = 25$.

In this problem, 4 corresponds to c, $7x-3$ corresponds to $f(x)$, and 25 corresponds to L in the limit definition.

I have to show that, given any $\epsilon > 0$, there is a δ, such that

$$\text{if} \quad \delta > |x-4| > 0, \quad \text{then} \quad \epsilon > |(7x-3)-25|.$$

Notice that I'm *given* ϵ, but I'm not told its value (which was the case in earlier examples). All I can assume is that it's some positive number. I have to come up with a δ that meets the condition above. To do this, I work backwards as I did in earlier examples. This is "scratchwork", and doesn't count as the "real" proof, which will come afterward.

Scratchwork. I want $\epsilon > |(7x-3)-25|$. I'll work backwards from this and try to get something that looks like "(whatever) $> |x-4|$". Then I'll set $\delta =$ (whatever) and try to do the real proof.

$$\begin{aligned} \epsilon &> |(7x-3)-25| \\ \epsilon &> |7x-28| \\ \epsilon &> 7|x-4| \\ \frac{\epsilon}{7} &> |x-4| \end{aligned}$$

Okay — I'll try $\delta = \dfrac{\epsilon}{7}$.

The real proof. Let $\delta = \dfrac{\epsilon}{7}$. I must show that

$$\text{if} \quad \delta > |x-4| > 0, \quad \text{then} \quad \epsilon > |(7x-3)-25|.$$

When you are proving an "if-then" statement, you get to *assume* the "if" part, and you *prove* the "then" part. So assume

$$\frac{\epsilon}{7} = \delta > |x-4|.$$

The rest of the proof is easy: I just reverse the steps I did on scratchwork:

$$\begin{aligned} \frac{\epsilon}{7} &> |x-4| \\ \epsilon &> 7|x-4| \\ \epsilon &> |7x-28| \\ \epsilon &> |(7x-3)-25| \end{aligned}$$

Therefore, by the limit definition,

$$\lim_{x\to 4}(7x-3)=25. \quad \Box$$

A similar approach works for limits of the form $\lim\limits_{x\to c}(ax+b)$. Here is a harder example.

Example. Prove that $\lim\limits_{x\to 3}(x^2+5x)=24$.

In this case, 3 corresponds to c, x^2+5x corresponds to $f(x)$, and 24 corresponds to L.

Scratchwork. I want $\epsilon > |(x^2+5x)-24|$. I'll work backwards from this and try to get something that looks like "(whatever) $> |x-3|$". Then I'll set $\delta =$ (whatever) and try to do the real proof.

$$\begin{aligned}\epsilon &> |(x^2+5x)-24|\\ \epsilon &> |x^2+5x-24|\\ \epsilon &> |(x+8)(x-3)|\\ \epsilon &> |x+8||x-3|\end{aligned}$$

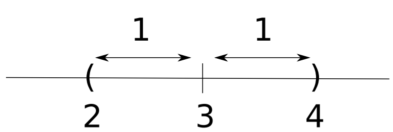
I can't just divide both sides by $|x+8|$ (like I divided by 7 in the last example:

$$\frac{\epsilon}{|x+8|} > |x-3|.$$

The problem is that I can't set $\delta = \dfrac{\epsilon}{|x+8|}$, because I would need to know x in order to know δ — but δ is supposed to determine the range of x's.

Instead, I need to make a "preliminary" setting of δ. I'll provisionally set $\delta = 1$. Then

$$\begin{aligned}1 = \delta &> |x-3|\\ 2 < x &< 4\end{aligned}$$



Remember that you have *complete control* over δ. Setting δ to 1 is like adjusting a setting on an instrument, where you make an initial rough setting, then fine-tune it. We'll see how this works out when we write the "real proof".

Adding 8 to each term, I get

$$\begin{aligned}2 < x &< 4\\ 10 < x+8 &< 12\\ |x+8| &< 12\end{aligned}$$

Remember that I want the inequality $\epsilon > |x+8||x-3|$.

If I could get $\epsilon > 12|x-3|$, then I'd have

$$\begin{aligned}12 &> |x+8|\\ 12|x-3| &> |x+8||x-3|\\ \epsilon > 12|x-3| &> |x+8||x-3|\end{aligned}$$

But

$$\begin{aligned} \epsilon &> 12|x-3| \\ \frac{\epsilon}{12} &> |x-3| \end{aligned}$$

It looks like I should try $\delta = \dfrac{\epsilon}{12}$ ... but then, I remember I needed to set $\delta = 1$ earlier. How can I get *both* of these things to happen? The idea is to make δ the *smaller* of the two numbers 1 and $\dfrac{\epsilon}{12}$ — in symbols,

$$\delta = \min\left(1, \frac{\epsilon}{12}\right).$$

("min" stands for "minimum".) This means that

$$1 \ge \delta \quad \text{and} \quad \frac{\epsilon}{12} \ge \delta.$$

The real proof. Let $\delta = \min\left(1, \dfrac{\epsilon}{12}\right)$. I must show that:

$$\text{if} \quad \delta > |x-3| > 0, \quad \text{then} \quad \epsilon > |(x^2+5x)-24|.$$

So I may *assume* $\delta > |x-3| > 0$, and I have to *prove* $\epsilon > |(x^2+5x)-24|$.
As I noted in the scratchwork, I know that

$$1 \ge \delta \quad \text{and} \quad \frac{\epsilon}{12} \ge \delta.$$

Take $1 \ge \delta$ first. Then

$$\begin{aligned} 1 \ge \delta &> |x-3| \\ 4 &> x > 2 \\ 12 &> x+8 > 10 \\ 12 &> |x+8| \end{aligned}$$

Next, I'll use $\dfrac{\epsilon}{12} \ge \delta$. Multiply this inequality and the inequality $12 > |x+8|$ to get

$$\begin{aligned} \epsilon = 12 \cdot \frac{\epsilon}{12} &> |x+8||x-3| \\ \epsilon &> |x^2+5x-24| \\ \epsilon &> |(x^2+5x)-24| \end{aligned}$$

Therefore, I've proved that $\lim_{x \to 3}(x^2+5x) = 24$. $\square$
