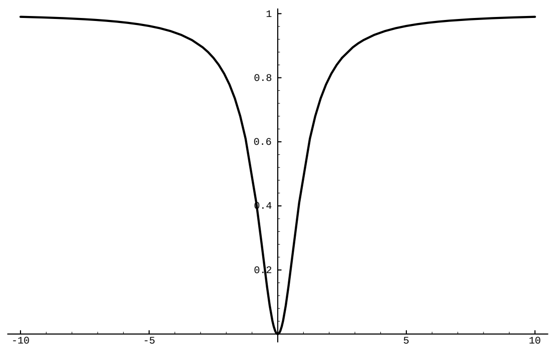
Limits at Infinity

In this section, I'll discuss the limit of a function $f(x)$ as x goes to ∞ and $-\infty$. We'll see that this is related to **horizontal asyptotes** of a graph.

It's natural to discuss **vertical asymptotes** as well, and I'll explain how these are connected to values of x where the limit of $f(x)$ becomes infinite.

Let's start with an example. Here is the graph of $f(x) = \dfrac{x^2}{x^2+1}$:



The graph approaches the horizontal line $y = 1$ as it goes out to the left and right. You write:

$$\lim_{x\to+\infty} \frac{x^2}{x^2+1} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{x^2}{x^2+1} = 1.$$

Here's a rough definition. If the graph of $f(x)$ approaches $y = L$ as you plug in larger and larger *positive* values for x, then

$$\lim_{x\to+\infty} f(x) = L.$$

Likewise, if the graph of $f(x)$ approaches $y = L$ as you plug in larger and larger *negative* values for x, then

$$\lim_{x\to-\infty} f(x) = L.$$

As a numerical example, consider $f(x) = \dfrac{x^2}{x^2+1}$. If you set $x = 10^6$, you get

$$f(x) \approx 0.999999999999000000000000999999999999000000000001.$$

That's pretty close to 1, isn't it?

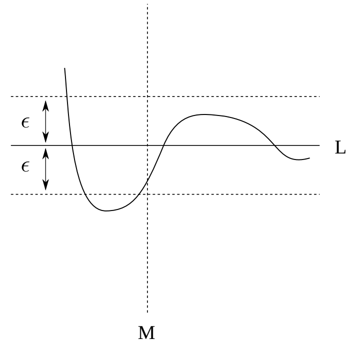
Here are the precise definitions. They're analogous to the ϵ-δ definitions of ordinary limits.

Definition. $\displaystyle\lim_{x\to\infty} f(x) = L$ means: For every $\epsilon > 0$, there is a number M, such that:

$$\text{If} \quad x > M, \quad \text{then} \quad \epsilon > |f(x) - L|.$$

(You can give a similar definition for $\displaystyle\lim_{x\to-\infty} f(x) = L$.)

The definition says that I can make $f(x)$ as close to L as I want, by making x sufficiently large.



As the picture shows, values of x greater than M produce values of $f(x)$ that lie within ϵ of L.

Example. Prove that $\displaystyle\lim_{x\to\infty} \frac{10x+4}{5x+1} = 2$.

Scratch work. I'll start by working backwards from ϵ to M.

$$\epsilon > \left|\frac{10x+4}{5x+1} - 2\right| = \left|\frac{(10x+4) - 2(5x+1)}{5x+1}\right| = \left|\frac{2}{5x+1}\right| = \frac{2}{5x+1}.$$

(I can remove the absolute value bars, since $x \to \infty$ means x will be large and positive.)
So

$$\begin{aligned} 5x+1 &> \frac{2}{\epsilon} \\ 5x &> \frac{2}{\epsilon} - 1 \\ x &> \frac{1}{5}\left(\frac{2}{\epsilon} - 1\right) \end{aligned}$$

This suggests that I should take $M = \dfrac{1}{5}\left(\dfrac{2}{\epsilon} - 1\right)$.

The reason for doing things this way is that you may not prove something by assuming what you want to prove. So the "working backward" part isn't by itself a valid proof: It is possible that some of the steps aren't reversible. You can ensure that everything works properly by writing the proof in the correct order, from assumptions to conclusion.

The real proof. Let $\epsilon > 0$. Take $M = \dfrac{1}{5}\left(\dfrac{2}{\epsilon} - 1\right)$.

Then if $x > M$, I have

$$\begin{aligned} x &> \frac{1}{5}\left(\frac{2}{\epsilon} - 1\right) \\ 5x &> \frac{2}{\epsilon} - 1 \\ 5x+1 &> \frac{2}{\epsilon} \end{aligned}$$

Note that since $\epsilon > 0$, the last inequality implies $5x + 1 > 0$. So

$$\epsilon > \frac{2}{5x+1}$$
$$\epsilon > \left|\frac{2}{5x+1}\right|$$

Dividing by $5x+1$ in the first step is okay, because $5x+1 > 0$ (so the inequality doesn't "flip"). Likewise, the second step is okay, because $5x + 1 > 0$, so $\dfrac{2}{5x+1}$ is positive, so I can add the absolute values.

Continuing, I have

$$\epsilon > \left|\frac{2}{5x+1}\right| = \left|\frac{10x+4}{5x+1} - 2\right|.$$

This shows that $\displaystyle \lim_{x\to\infty} \frac{10x+4}{5x+1} = 2$. □

Most of the properties of ordinary limits hold for limits as $x \to \pm\infty$.

Theorem. (a)

$$\lim_{x\to\infty} (f(x) + g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x).$$

(b) If k is a number,

$$\lim_{x\to\infty} (k \cdot f(x)) = k \cdot \lim_{x\to\infty} f(x).$$

(c)

$$\lim_{x\to\infty} (f(x) \cdot g(x)) = (\lim_{x\to\infty} f(x)) \cdot (\lim_{x\to\infty} g(x)).$$

(d) If $\displaystyle \lim_{x\to\infty} g(x) \neq 0$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}.$$

The statements mean that if the limits on the right side of the equation are defined, then the limits on the left sides are defined, and the two sides are equal.

Proof. I'll prove (a) by way of example. As in most limit proofs, you discover what to do by working backward ("on scratch paper"). Then you write the "real proof" forward. I'll omit the scratch work in this case.

A reminder about something before I start: I'll use the **Triangle Inequality**, which says that if p and q are real numbers, then

$$|p| + |q| \geq |p + q|.$$

Suppose that

$$\lim_{x\to\infty} f(x) = A \quad \text{and} \quad \lim_{x\to\infty} g(x) = B.$$

I want to show that

$$\lim_{x\to\infty} (f(x) + g(x)) = A + B.$$

Let $\epsilon > 0$.

Since $\displaystyle \lim_{x\to\infty} f(x) = A$, I can find a number M such that if $x > M$, then

$$\frac{1}{2}\epsilon > |f(x) - A|.$$

Since $\displaystyle \lim_{x\to\infty} g(x) = A$, I can find a number N such that if $x > N$, then

$$\frac{1}{2}\epsilon > |g(x) - B|.$$

Suppose that $x > \max(M, N)$. This means that $x > M$ and $x > N$, so both of the $\dfrac{1}{2}\epsilon$ inequalities hold. Hence, adding the inequalities, I get

$$\begin{aligned} \epsilon &= \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \\ &> |f(x) - A| + |g(x) - B| \\ &\geq |\,(f(x) - A) + (g(x) - B)\,| \\ &= |\,(f(x) + g(x)) - (A + B)\,| \end{aligned}$$

(I used the Triangle Inequality in the "$\geq$" step.) This proves that

$$\lim_{x\to\infty} (f(x) + g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x).$$

Similar ideas are used in the proofs of (b), (c), and (d), though in some cases the algebra involved is a little trickier. $\square$

Here is a property that I'll use frequently.

Proposition. Let $k > 0$. Then

$$\lim_{x\to+\infty} \frac{1}{x^k} = 0.$$

Proof. Let $\epsilon > 0$. I must find a number M such that if $x > M$ and $\dfrac{1}{x^k}$ is defined, then

$$\epsilon > \left|\frac{1}{x^k} - 0\right| = \left|\frac{1}{x^k}\right|.$$

Set $M = \dfrac{1}{\epsilon^{1/k}}$. Note that $\epsilon^{1/k}$ is defined and positive, since $\epsilon > 0$ and $k > 0$. Suppose $x > M$. Since M is positive, so is x, so $\dfrac{1}{x^k}$ is defined and positive.

I have

$$\begin{aligned} x > M &= \frac{1}{\epsilon^{1/k}} \\ x^k &> \frac{1}{\epsilon} \\ \epsilon &> \frac{1}{x^k} \\ \epsilon &> \frac{1}{x^k} \\ \epsilon &> \left|\frac{1}{x^k}\right| \end{aligned}$$

Hence, $\displaystyle\lim_{x\to+\infty} \frac{1}{x^k} = 0$. $\square$

Is it true that

$$\lim_{x\to-\infty} \frac{1}{x^k} = 0\ ?$$

It is — provided that $\dfrac{1}{x^k}$ is defined. What could go wrong? Suppose $k = \dfrac{1}{2}$. Then $\displaystyle\lim_{x\to-\infty} \frac{1}{x^{1/2}}$ is undefined, since $x^{1/2}$ is not defined if x is negative and $x \to -\infty$ means that x is taking on negative values. On the other hand,

$$\lim_{x\to-\infty} \frac{1}{x^4} = 0.$$

Here are some examples of limits at $+\infty$ and $-\infty$.

Example. (a) Compute $\lim_{x\to+\infty} \dfrac{2x^3 - 3x + 7}{4 - x^2 - 5x^3}$.

(b) Compute $\lim_{x\to-\infty} \dfrac{x^{15} - 3x^9 + 47}{x^2 - x + 1}$.

(c) Compute $\lim_{x\to+\infty} \dfrac{x - 17}{x^{3/2} - 4x + 2} = 0$.

(a) In limits at infinity involving powers of x, the rule of thumb is that the biggest powers dominate. In this case, the biggest powers on the top and bottom are the x^3's. Therefore, the limit in (a) behaves almost like

$$\lim_{x\to+\infty} \frac{2x^3}{-5x^3} = -\frac{2}{5}.$$

So you expect the answer to be $-\dfrac{2}{5}$.

On way to see this formally is to divide the top and bottom by x^3:

$$\lim_{x\to+\infty} \frac{2x^3 - 3x + 7}{4 - x^2 - 5x^3} = \lim_{x\to+\infty} \frac{2 - \dfrac{3}{x^2} + \dfrac{7}{x^3}}{\dfrac{4}{x^3} - \dfrac{1}{x} - 5}.$$

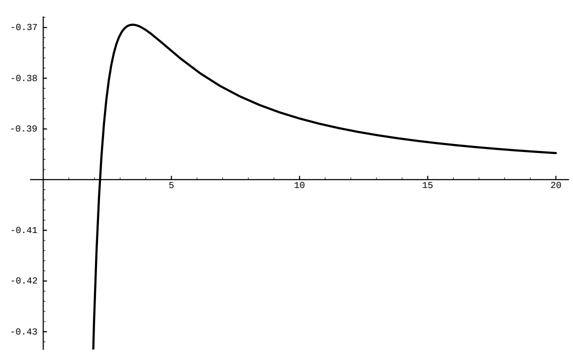
Now as $x \to +\infty$,

$$\frac{\text{a number}}{x^{\text{positive power}}} \to \frac{\text{a number}}{\text{something big}} = 0.$$

Hence,

$$\lim_{x\to+\infty} \frac{2 - \dfrac{3}{x^2} + \dfrac{7}{x^3}}{\dfrac{4}{x^3} - \dfrac{1}{x} - 5} = \lim_{x\to+\infty} \frac{2 - 0 + 0}{0 - 0 - 5} = -\frac{2}{5}.$$

Here's a picture of $\dfrac{2x^3 - 3x + 7}{4 - x^2 - 5x^3}$:



□

(b)

$$\lim_{x\to-\infty} \frac{x^{15} - 3x^9 + 47}{x^2 - x + 1} = -\infty.$$

In this case, the x^{15} on top beats out the puny x^2 on the bottom.

By the way, it would be correct to say this limit *diverges*. However, it's more informative to say *how* it diverges. In this case, the function $\dfrac{x^{15} - 3x^9 + 47}{x^2 - x + 1}$ becomes large and negative, so you write $-\infty$ for the limit. □

(c)

$$\lim_{x \to +\infty} \frac{x - 17}{x^{3/2} - 4x + 2} = 0.$$

Here the $x^{3/2}$ on the bottom beats out the x^1 on the top. □

Suppose that

$$\lim_{x \to +\infty} f(x) = L.$$

I noted above that this means that the graph of $f(x)$ approaches the line $y = L$ as you move to the right.

Likewise, suppose

$$\lim_{x \to -\infty} f(x) = L.$$

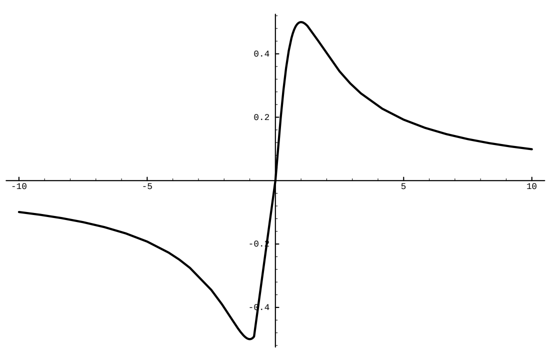
This means that the graph of $f(x)$ approaches the line $y = L$ as you move to the left. In these situations, $y = L$ is a **horizontal asymptote** for the graph of $f(x)$.

Not all graphs have horizontal asymptotes — for example, $y = x^2$ goes to ∞ as $x \to \infty$ and as $x \to -\infty$. You can check for the presence of horizontal asymptotes by computing $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ and seeing if either is a number.

Example. Find the horizontal asymptotes (if any) of $y = \dfrac{x}{x^2 + 1}$.

$$\lim_{x \to \infty} \frac{x}{x^2 + 1} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{x}{x^2 + 1} = 0.$$

Therefore, $y = 0$ is a horizontal asymptote for the graph at $+\infty$ and at $-\infty$. The graph is shown below:



□

Example. Find the horizontal asymptotes of $f(x) = \dfrac{x}{\sqrt{x^2 + 4}}$.

The limit at $+\infty$ works without any surprises. The highest power on the top and the bottom is x (since $\sqrt{x^2}$ looks like x), so divide the top and bottom by x:

$$\lim_{x \to +\infty} \frac{x}{\sqrt{x^2 + 4}} = \lim_{x \to +\infty} \frac{1}{\dfrac{1}{x}\sqrt{x^2 + 4}} = \lim_{x \to +\infty} \frac{1}{\sqrt{1 + \dfrac{4}{x^2}}} = \frac{1}{1} = 1.$$

However, the limit at $-\infty$ is a little tricky! Here's the computation:

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+4}} = \lim_{x\to-\infty} \frac{1}{\dfrac{1}{x}\sqrt{x^2+4}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\dfrac{4}{x^2}}} = \frac{1}{-1} = -1.$$

Where did that negative sign come from? Look at the bottom, which was $\dfrac{1}{x}\sqrt{x^2+4}$. x is going to $-\infty$, so x is taking on *negative* values. Now $\sqrt{\cdot}$ is positive, so $\dfrac{1}{x}\sqrt{x^2+4}$ is *negative*.

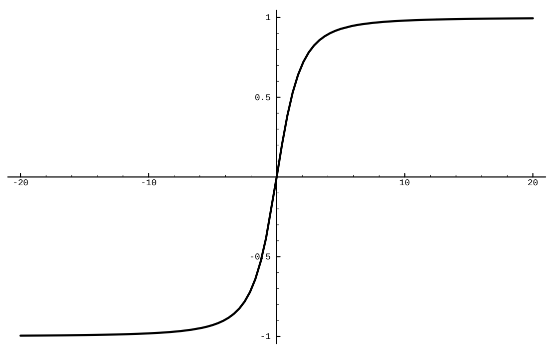
When you push the $\dfrac{1}{x}$ into the square root, you must leave a negative sign outside. Otherwise, you'd have $\sqrt{\text{junk}}$, a *positive* thing.

Alternatively, to think of it the other way,

$$\sqrt{x^2} = |x|.$$

So if x is negative (because $x \to -\infty$), I have $\sqrt{x^2} = |x| = -x$.

Thus, this is a case where it matters that x is going to $-\infty$, as opposed to $+\infty$. Here's the graph:



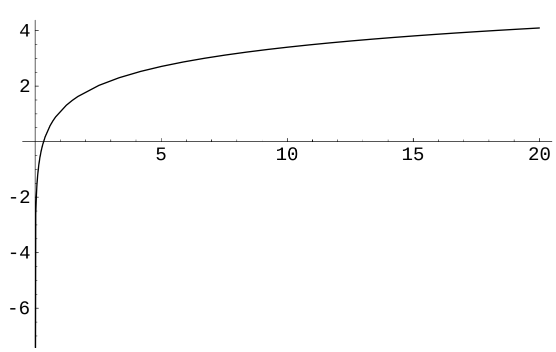
$\square$

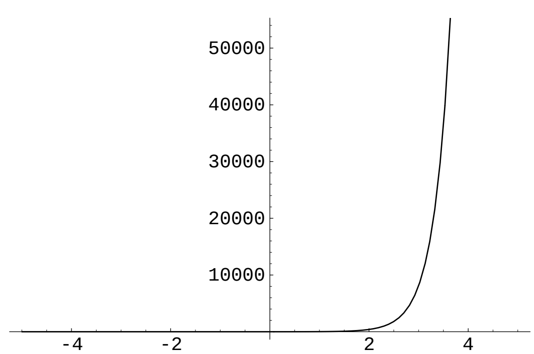
How do logarithms and exponentials behave as $x \to +\infty$ or $x \to -\infty$? The relevant facts are summarized below.

$$\lim_{x\to+\infty} \ln ax = +\infty \quad \text{and} \quad \lim_{x\to0^+} \ln ax = -\infty \quad \text{if} \quad a > 0.$$

$$\lim_{x\to+\infty} e^{ax} = +\infty \quad \text{and} \quad \lim_{x\to-\infty} e^{ax} = 0 \quad \text{if} \quad a > 0.$$

I've graphed $y = \ln 2x$ (on the left) and $y = e^{3x}$ (on the right) below; you can see that the pictures are consistent with the formulas above.





For example, the graph of $y = \ln 2x$ goes downward asymptotically along the y-axis from the right. This confirms that $\lim_{x \to 0^+} \ln 2x = -\infty$.

Likewise, the graph of e^{3x} rises sharply as you go to the right; this confirms that $\lim_{x \to +\infty} e^{3x} = +\infty$.

Note that if $a < 0$ in e^{ax}, the limits are reversed. Specifically,

$$\lim_{x \to +\infty} e^{ax} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^{ax} = +\infty \quad \text{if} \quad a < 0.$$

Example. (a) Compute

$$\lim_{x \to +\infty} \ln 1.37x \quad \text{and} \quad \lim_{x \to 0^+} \ln 1.37x.$$

(b) Compute

$$\lim_{x \to +\infty} e^{6x} \quad \text{and} \quad \lim_{x \to -\infty} e^{6x}.$$

(c) Compute

$$\lim_{x \to +\infty} e^{-\sqrt{2}x} \quad \text{and} \quad \lim_{x \to -\infty} e^{-\sqrt{2}x}.$$

(a)

$$\lim_{x \to +\infty} \ln 1.37x = +\infty \quad \text{and} \quad \lim_{x \to 0^+} \ln 1.37x = -\infty.$$

(b)

$$\lim_{x \to +\infty} e^{6x} = +\infty \quad \text{and} \quad \lim_{x \to -\infty} e^{6x} = 0.$$

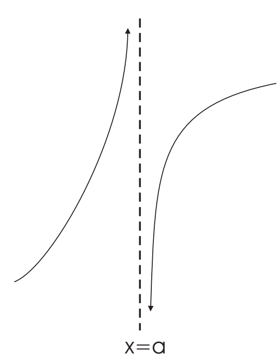
(c)

$$\lim_{x \to +\infty} e^{-\sqrt{2}x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^{-\sqrt{2}x} = +\infty. \quad \square$$

Infinity can also appear in limits in connection with **vertical asymptotes**. I'll say that the graph of a function $y = f(x)$ has a **vertical asymptote** at $x = a$ if at least one of the limits

$$\lim_{x \to a^+} f(x) \quad \text{or} \quad \lim_{x \to a^-} f(x) \quad \text{is either} \quad +\infty \quad \text{or} \quad -\infty.$$

Example. The graph below has a vertical asymptote at $x = a$:



What are $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$?

$$\lim_{x\to a^+} f(x) = -\infty \quad \text{while} \quad \lim_{x\to a^-} f(x) = +\infty. \quad \square$$

In general, you might *suspect* the presence of a vertical asymptote at an *isolated* value of x for which $f(x)$ is undefined. To **confirm** your suspicion, you need to compute the left- and right-hand limits at the point.

Example. Locate the vertical asymptotes of $f(x) = \dfrac{1}{(x-1)(x-2)}$ and sketch the graph near the asymptotes.

$f(x)$ is undefined at $x = 1$ and at $x = 2$. I'll check for vertical asymptotes by computing the left- and right-hand limits at $x = 1$ and at $x = 2$. I'll work through the first one carefully.

$$\lim_{x\to 1^+} \frac{1}{(x-1)(x-2)} = -\infty.$$

To see this, consider numbers close to 1 but to the right of 1. Then $x - 1$ will be positive, while $x - 2$ will be negative. For example, if $x = 1.1$, then $x - 1 = 0.1$ while $x - 2 = -0.9$. All together, the fraction $\dfrac{1}{(x-1)(x-2)}$ will be negative. But plugging $x = 1$ into the fraction gives $\dfrac{1}{0}$. Since the result is negative and infinite, it must be $-\infty$.

You can see numerical evidence for this by plugging (e.g.) $x = 1.001$ into $\dfrac{1}{(x-1)(x-2)}$.

$$\frac{1}{(1.001-1)(1.001-2)} \approx -1001.$$

This is a large negative number, which suggests that the limit is $-\infty$.

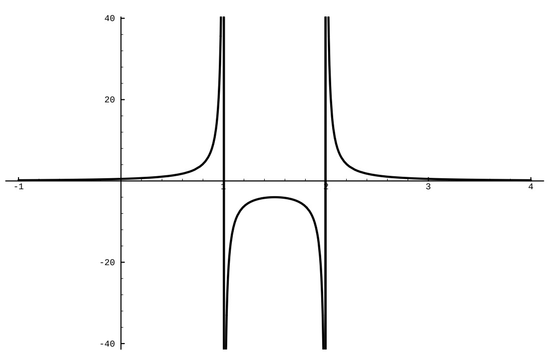
In similar fashion,

$$\lim_{x\to 1^-} \frac{1}{(x-1)(x-2)} = +\infty,$$

$$\lim_{x\to 2^+} \frac{1}{(x-1)(x-2)} = +\infty,$$

$$\lim_{x\to 2^-} \frac{1}{(x-1)(x-2)} = -\infty.$$

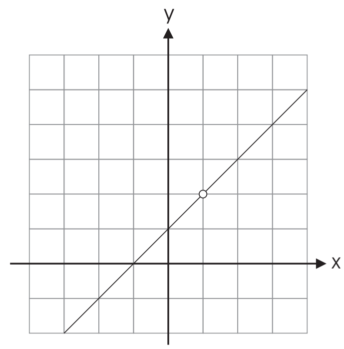
Here's the graph:



$\square$

Example. $f(x) = \dfrac{x^2 - 1}{x - 1}$ is undefined at $x = 1$. Does it have a vertical asymptote at $x = 1$?

The fact that a function is undefined at an isolated value does not imply that it has a vertical asymptote there. The graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$ looks like this:



You can see this by noting that, for $x \neq 1$,

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1} = x + 1.$$

Thus, the graph is the same as the graph of the line $y = x + 1$ except at $x = 1$, where there's a hole. In other words,

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2.$$

In particular, the graph does not have a vertical asymptote at $x = 1$. $\square$
