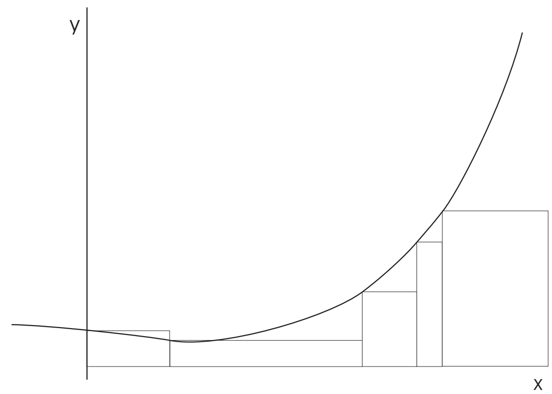
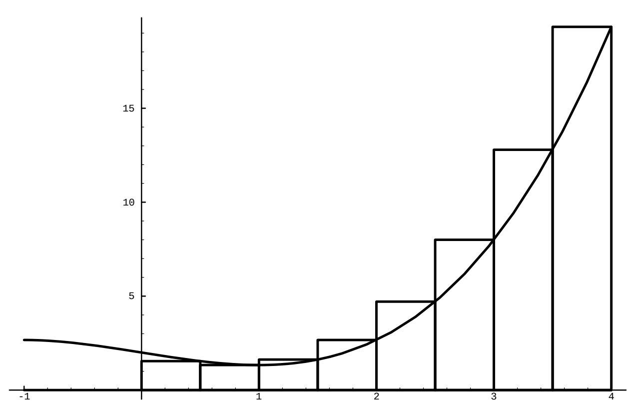
Rectangle Sums

You can approximate the area under a curve using rectangles. To do this, divide the base interval into pieces (**subintervals**). Then on each subinterval, build a rectangle that goes up to the curve.



What does it mean to "go up to the curve"? You have to make a choice about how the height of each rectangle depends on the curve. In the picture above, for my rectangle height I always used the height of the curve above the left-hand endpoint of each subinterval.

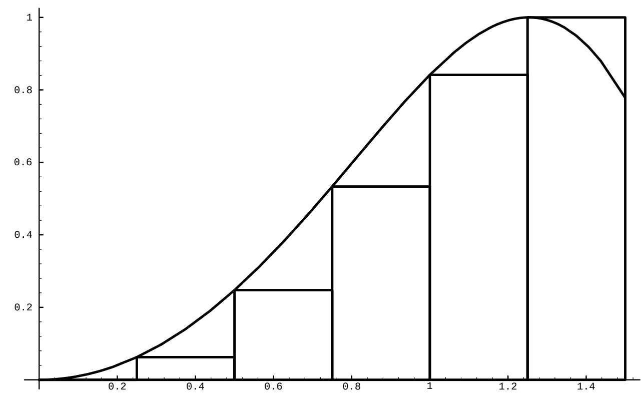
In the picture above, I used subintervals of different sizes. For simplicity, you will often use subintervals of the same size — so your rectangles all have the same width. In the picture below, I've used 8 rectangles of equal widths, and for my rectangle height I always used the height of the curve above the right-hand endpoint of each subinterval.



Here's an example with a specific function. I'll use $f(x) = \sin(x^2)$. In each case, I'll use the base interval $0 \leq x \leq 1.5$ divided into 6 equal subintervals:

$$[0, 0.25], [0.25, 0.5], [0.5, 0.75], [0.75, 1.0], [1.0, 1.25], [1.25, 1.5]$$

Here is the picture if I use the left-hand endpoint of each subinterval to get the height of each rectangle:



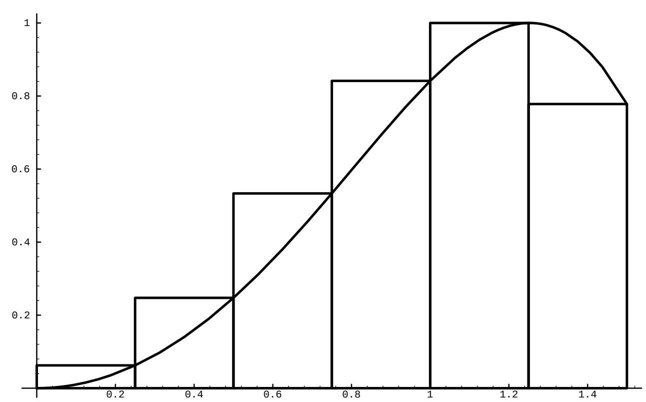
The sum of the areas of the rectangles is:

$$(\sin 0)(0.25) + \left(\sin \frac{1}{16}\right)(0.25) + \left(\sin \frac{1}{4}\right)(0.25) + \left(\sin \frac{9}{16}\right)(0.25) + (\sin 1)(0.25) + \left(\sin \frac{25}{16}\right)(0.25) =$$

$$\left(\sin 0 + \sin \frac{1}{16} + \sin \frac{1}{4} + \sin \frac{9}{16} + \sin 1 + \sin \frac{25}{16}\right)(0.25) \approx 0.671151.$$

Notice that the rectangle width 0.25 factors out of the sum — you add up the f's, then multiply by 0.25. This will always be possible if you use subintervals of equal length.

Here is the picture if I use the right-hand endpoint of each subinterval to get the height of each rectangle:

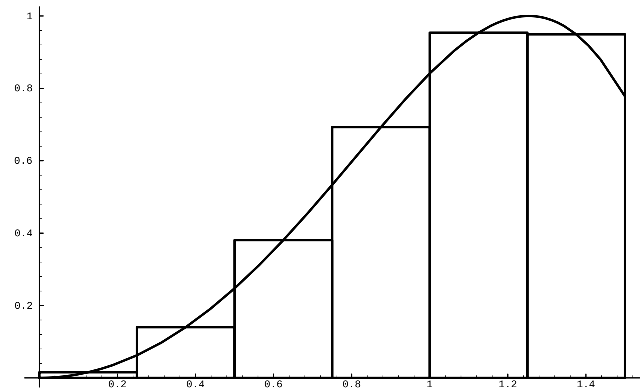


In this case, the sum of the areas of the rectangles is

$$\left(\sin \frac{1}{16}\right)(0.25) + \left(\sin \frac{1}{4}\right)(0.25) + \left(\sin \frac{9}{16}\right)(0.25) + (\sin 1)(0.25) + \left(\sin \frac{25}{16}\right)(0.25) + \left(\sin \frac{9}{4}\right)(0.25) =$$

$$\left(\sin \frac{1}{16} + \sin \frac{1}{4} + \sin \frac{9}{16} + \sin 1 + \sin \frac{25}{16} + \sin \frac{9}{4}\right)(0.25) \approx 0.865669.$$

Here is the picture if I use the midpoint of each subinterval to get the height of each rectangle:



The midpoints of the subintervals are:

$$0.125, 0.375, 0.625, 0.875.1.125, 1.375$$

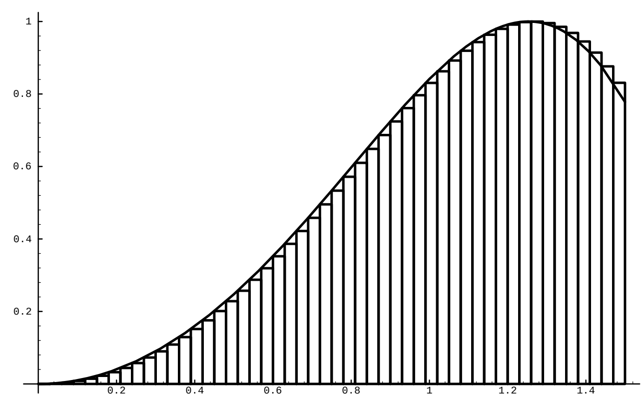
In this case, the sum of the areas of the rectangles is

$$\left(\sin\frac{1}{64}\right)(0.25)+\left(\sin\frac{9}{64}\right)(0.25)+\left(\sin\frac{25}{64}\right)(0.25)+\left(\sin\frac{49}{64}\right)(0.25)+\left(\sin\frac{81}{64}\right)(0.25)+\left(\sin\frac{121}{64}\right)(0.25) =$$

$$\left(\sin\frac{1}{64}+\sin\frac{9}{64}+\sin\frac{25}{64}+\sin\frac{49}{64}+\sin\frac{81}{64}+\sin\frac{121}{64}\right)(0.25) \approx 0.783156.$$

By comparison, the actual area under the curve is around 0.778238.

You can get better approximations by taking more rectangles. For example, here is the left-hand endpoint picture with 50 rectangles:



Notice how much better the rectangles approximate the area under the curve.

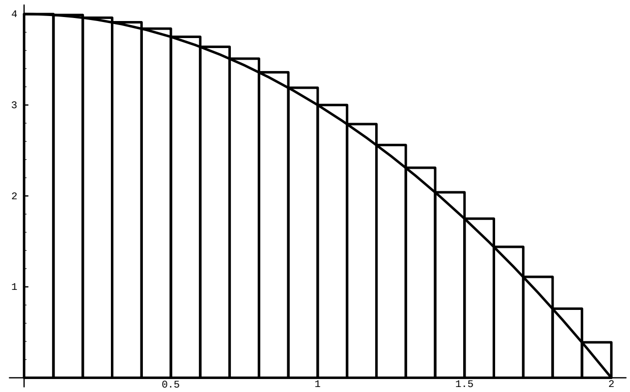
With 200 rectangles, the left-hand endpoint sum is 0.775311, the right-hand endpoint sum is 0.781147, and the midpoint sum is 0.778242. The three values are close to the actual value 0.778238.

Example. Approximate the area under $f(x) = 4 - x^2$ for $0 \le x \le 2$, using 20 *circumscribed* rectangles of equal width.

Circumscribed means that you should use the *largest* function value on each interval to get the height of a rectangle. My subintervals are

$$[0, 0.1], [0.1, 0.2], [0.2, 0.3], \ldots, [1.9, 2.0].$$

In general, it can be difficult to determine where the largest function value is. However, by graphing $f(x) = 4 - x^2$ on $0 \leq x \leq 2$, you can see that the largest function value for each subinterval occurs at the left-hand endpoint.



So I use

$$0 \quad \text{for the first subinterval,}$$

$$0.1 \quad \text{for the second subinterval,}$$

$$0.2 \quad \text{for the third subinterval.}$$

I continue in this fashion, all the way up to

$$1.9 \quad \text{for the twentieth subinterval.}$$

I can write these points as

$$0.1n, \quad \text{for} \quad n = 0, 1, \ldots, 19.$$

So my function values are

$$f(0), f(0.1), f(0.2), \ldots, f(1.9).$$

These are the rectangles heights. Each height is multiplied by a width of 0.1. The total is

$$f(0) \cdot 0.1 + f(0.1) \cdot 0.1 + f(0.2) \cdot 0.1 + \cdots + f(1.9) \cdot 0.1 = \sum_{n=0}^{19} f(0.1n) \cdot 0.1 =$$

$$\sum_{n=0}^{19} \left(4 - (0.1n)^2\right) \cdot 0.1 = \sum_{n=0}^{19} (0.4 - 0.001n^2).$$

Now the sum is in a form you can evaluate on your calculator. You should get 5.53; the actual value is $5.33333\ldots$. □

Example. Approximate the area under $y = \dfrac{1}{1 + x^3}$ from $x = 0$ to $x = 1$ using 20 equal subintervals and evaluating the function at the left-hand endpoints.

$$\sum_{n=0}^{19} f\left(n \cdot \frac{1}{20}\right) \cdot \frac{1}{20} = \sum_{n=0}^{19} \frac{1}{1 + \left(\dfrac{n}{20}\right)^3} \cdot \frac{1}{20} = 400 \sum_{n=0}^{19} \frac{1}{8000 + n^3}.$$

You can use a calculator to approximate this sum; it's around 0.84799. □

In the preceding examples, I've assumed that the subintervals (which give the widths of the rectangle) are the same size. I've also chosen the evaluation points systematically — left-hand endpoints, right-hand endpoints, midpoints. These are conveniences to make setting up the computations simple.

In general, the subintervals don't have to be the same size, and I don't have to choose the evaluation points systematically. For example, here is an approximation to the area under $y = x^2$ from $x = 0$ to $x = 6$.

interval	x	$f(x)$	$f(x) \cdot \Delta x$
$[0, 2]$	1.0	1.0	2.0
$[2, 3]$	2.8	7.84	7.84
$[3, 3.5]$	3.0	9.0	4.5
$[3.5, 6]$	5.0	25.0	37.5
sum			51.84

This gives an approximate area of 51.84. The actual area is 77.