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Separation of Variables

Separation of variables is a method for solving a **differential equation**. I'll illustrate with some examples.

Example. Solve $\dfrac{dy}{dx} = 2xy$.

"Solve" usually means to find y in terms of x. In general, I'll be satisfied if I can eliminate the derivative by integration.

First, I rearrange the equation to get the x's on one side and the y's on the other (*separation*):

$$\frac{dy}{y} = 2x\,dx.$$

This is a *formal* manipulation, since I'm temporarily treating $\dfrac{dy}{dx}$ as a quotient of dy by dx. (See the remark below.)

Next, I *integrate* both sides:

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln|y| = x^2 + C$$

I only need an arbitrary constant on one side of the equation. Finally, I *solve* for y in terms of x, if possible:

$$e^{\ln|y|} = e^{x^2+C}$$

$$|y| = e^C e^{x^2}$$

Here's a convenient trick which I'll use in these situations. Think of $|y|$ as $\pm y$. Move the $\pm$ to the other side:

$$y = \mp e^C e^{x^2}.$$

Now *define* $C_0 = \mp e^C$:

$$y = C_0 e^{x^2}.$$

The last step makes the equation nicer, and it's easier to solve for the arbitrary constant when you have an *initial value problem*. □

Remark. Here's a justification for the formal manipulation with dx and dy. Think of x and y as depending on a third variables t, so $x = f(t)$ and $y = g(t)$. By the Chain Rule,

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$$

The initial equation becomes

$$\frac{dy}{dx} = 2xy$$

$$\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = 2xy$$

$$y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

Then integrate both sides with respect to t.

$$\int y\frac{dy}{dt}\,dt = \int 2x\frac{dx}{dt}\,dt$$
$$\int y\,dy = \int 2x\,dx$$

Then continue as above. In the example that follows, I'll just work formally with dx and dy.

Example. Solve $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{1}{y}$, where $y(2) = 4$.

Separate:

$$\frac{dy}{dx} = \frac{1}{y}(x+1)$$
$$y\,dy = (x+1)\,dx$$

Integrate:

$$\int y\,dy = \int (x+1)\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + C$$

In this case, solving would produce plus and minus square roots, so I'll leave the equation as is. Plug in the initial condition: When $x = 2$, $y = 4$:

$$\frac{1}{2}\cdot 4^2 = \frac{1}{2}\cdot 2^2 + 2 + C$$
$$8 = 2 + 2 + C$$
$$C = 4$$

Hence, the solution is

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + 4. \quad \square$$

I'll use separation of variables to solve the equations for **exponential growth** and **Neton's law of cooling**.