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Substitution

You can use **substitution** to convert a complicated integral into a simpler one. In these problems, I'll let u equal some convenient x-stuff — say $u = f(x)$. To complete the substitution, I must also substitute for dx. To do this, compute $\dfrac{du}{dx} = f'(x)$, so $du = f'(x)\,dx$. Then $dx = \dfrac{du}{f'(x)}$.

Example. Compute $\int (2x+3)^{100}\,dx$.

$$\int (2x+3)^{100}\,dx = \int u^{100}\cdot\frac{du}{2} = \frac{1}{2}\int u^{100}\,du = \frac{1}{202}u^{101} + C = \frac{1}{202}(2x+3)^{101} + C.$$

$$\left[u = 2x+3, \quad du = 2\,dx, \quad dx = \frac{du}{2}\right] \quad \square$$

Here's what's going on. By the Chain Rule,

$$\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x).$$

By the definition of antiderivative,

$$\int f'(g(x))\cdot g'(x)\,dx = f(g(x)) + C.$$

Now if $u = g(x)$, I have

$$\int f'(u)\,du = f(u) + C = f(g(x)) + C.$$

So

$$\int f'(g(x))\cdot g'(x)\,dx = \int f'(u)\,du.$$

The manipulations with dx and du are just a convenient way of doing the substitution. These are *not* the same "dx" and "du" we used in discussing differentials.

Example. Compute $\int \dfrac{dx}{\sqrt{4-7x}}$.

$$\int \frac{dx}{\sqrt{4-7x}} = \int \frac{1}{\sqrt{u}}\cdot\left(-\frac{du}{7}\right) = -\frac{1}{7}\int u^{-1/2}\,du = -\frac{2}{7}u^{1/2} + C = -\frac{2}{7}(4-7x)^{1/2} + C.$$

$$\left[u = 4-7x, \quad du = -7\,dx, \quad dx = -\frac{du}{7}\right] \quad \square$$

Example. Later on, I'll derive the integration formula

$$\int \frac{dx}{x} = \ln|x| + C.$$

Use this formula to compute $\int \dfrac{1}{3x+1}\,dx$.

$$\int \frac{1}{3x+1}\,dx = \int \frac{1}{u}\cdot\frac{du}{3} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x+1| + C.$$

$$\left[u = 3x+1, \quad du = 3\,dx, \quad dx = \frac{du}{3}\right] \quad \square$$

Example. Compute $\int x(x^2+5)^{50}\,dx$.

$$\int x(x^2+5)^{50}\,dx = \int xu^{50}\cdot\frac{du}{2x} = \frac{1}{2}\int u^{50}\,du = \frac{1}{102}u^{51} + C = \frac{1}{102}(x^2+5)^{51} + C.$$

$$\left[u = x^2+5, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}\right] \quad \square$$

Notice that in the second step in the last example, the x's cancelled out, leaving only u's. If the x's had failed to cancel, I wouldn't have been able to complete the substitution.

But what made the x's cancel? It was the fact that I got an x from the derivative of $u = x^2+5$. This leads to the following rule of thumb.

Substitute for something whose derivative is also there.

Example. Compute $\int (x+1)\sqrt{x^2+2x+5}\,dx$.

$$\int (x+1)\sqrt{x^2+2x+5}\,dx = \int (x+1)\sqrt{u}\cdot\frac{du}{2(x+1)} = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{2/3} + C = \frac{1}{3}(x^2+2x+5)^{3/2} + C.$$

$$\left[u = x^2+2x+5, \quad du = (2x+2)\,dx = 2(x+1)\,dx, \quad dx = \frac{du}{2(x+1)}\right] \quad \square$$

Example. Compute $\int \sin(3x+1)\,dx$.

$$\int \sin(3x+1)\,dx = \int \sin u\cdot\frac{du}{3} = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(3x+1) + C.$$

$$\left[u = 3x+1, \quad du = 3\,dx, \quad dx = \frac{du}{3}\right] \quad \square$$

Example. Compute $\int (\sin 5x)^7\cos 5x\,dx$.

$$\int (\sin 5x)^7\cos 5x\,dx = \int u^7\cos 5x\cdot\frac{du}{5\cos 5x} = \frac{1}{5}\int u^7\,du = \frac{1}{40}u^8 + C = \frac{1}{40}(\sin 5x)^8 + C.$$

$$\left[u = \sin 5x, \quad du = 5\cos 5x\,dx, \quad dx = \frac{du}{5\cos 5x}\right] \quad \square$$

Example. Compute $\displaystyle\int \frac{1}{\sqrt{x}(\sqrt{x}+9)^2}\,dx$.

$$\int \frac{1}{\sqrt{x}(\sqrt{x}+9)^2}\,dx = \int \frac{1}{\sqrt{x}u^2}\cdot 2\sqrt{x}\,du = 2\int u^{-2}\,du = -\frac{2}{u} + C = -\frac{2}{\sqrt{x}+9} + C.$$

$$\left[u = \sqrt{x}+9, \quad du = \frac{dx}{2\sqrt{x}}, \quad dx = 2\sqrt{x}\,du\right] \quad \square$$

Example. Compute $\displaystyle\int \frac{f'(x)}{(f(x)+3)^2}\,dx$.

$$\int \frac{f'(x)}{(f(x)+3)^2}\,dx = \int \frac{f'(x)}{u^2}\cdot\frac{du}{f'(x)} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{f(x)+3} + C.$$

$$\left[u = f(x)+3, \quad du = f'(x)\,dx, \quad dx = \frac{du}{f'(x)}\right] \quad \square$$

Example. Compute $\displaystyle\int \frac{\sin\frac{1}{x}}{x^2}\,dx$.

$$\int \frac{\sin\frac{1}{x}}{x^2}\,dx = \int \frac{\sin u}{x^2}\cdot(-x^2\,du) = -\int \sin u\,du = \cos u + C = \cos\frac{1}{x} + C.$$

$$\left[u = \frac{1}{x}, \quad du = -\frac{dx}{x^2}, \quad dx = -x^2\,du\right] \quad \square$$

The next problem introduces a new idea. In some cases, to replace the x's with u's, *you may need to solve the substitution equation for x.*

Example. Compute $\displaystyle\int (3x+4)(x+3)^{40}\,dx$.

There is no valid algebra which will allow me to multiply this out — unless I plan to multiply out $(x+3)^{40}$!

I'll let $u = x+3$, so $du = dx$. If I stopped with that, I'd have

$$\int (3x+4)(x+3)^{40}\,dx = \int (3x+4)u^{40}\,du.$$

I can't continue as-is, because I have both x's and u's in the integral.

To get rid of the x's, I solve the substitution equation $u = x+3$ for x, to get $x = u-3$. I can plug this into $3x+4$ to get everything in terms of u. Here's the work:

$$\int (3x+4)(x+3)^{40}\,dx = \int [3(u-3)+4]\,u^{40}\,du = \int (3u-5)u^{40}\,du = \int (3u^{41}-5u^{40})\,du =$$

$$[u = x+3, \quad du = dx, \quad x = u-3]$$

$$\frac{3}{42}u^{42} - \frac{5}{41}u^{41} + C = \frac{3}{42}(x+3)^{42} - \frac{5}{41}(x+3)^{41} + C. \quad \square$$

Example. Compute $\displaystyle \int \frac{4x+7}{\sqrt{x-2}}\,dx$.

In this problem, after making the substitution $u = x - 2$, I solve the substitution equation for x to get $x = u + 2$. Then I plug this into $4x + 7$ to get rid of the x's. Here's the work:

$$\int \frac{4x+7}{\sqrt{x-2}}\,dx = \int \frac{4(u+2)+7}{\sqrt{u}}\,du = \int \frac{4u+15}{\sqrt{u}}\,du = \int \left(4\sqrt{u} + \frac{15}{\sqrt{u}}\right)\,du =$$

$$\left[u = x - 2, \quad du = dx, \quad x = u + 2\right]$$

$$\frac{8}{3}u^{3/2} + 30u^{1/2} + C = \frac{8}{3}(x-2)^{3/2} + 30(x-2)^{1/2} + C. \quad \square$$
