Sums

Summation notation is used to denote a sum of terms. Usually, the terms follow a pattern or formula.

$$\sum_{k=1}^{n} f(k) \quad \text{is shorthand for} \quad f(0) + f(1) + \cdots + f(n).$$

In this case, 1 and n are the **limits of summation** and k is the **summation variable**. Each integer from the lower limit (in this case, 1) to the upper limit (in this case, n), *including both limits*, is substituted in turn for each occurence of k in the term $f(k)$. After that, the results are added.

You may use any variable for the summation variable, though most often people use the letters from i through n. The lower and upper limits may be any integers, as long as the upper limit is greater than or equal to the lower limit. The term in the sum may be any function of the summation variable; the summation variable may also be used as a subscript of superscript.

For example:

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55.$$

$$\sum_{n=1}^{4} 7 = 7 + 7 + 7 + 7 = 28.$$

$$\sum_{j=-2}^{2} \sqrt{b_j} = \sqrt{b_{-2}} + \sqrt{b_{-1}} + \sqrt{b_0} + \sqrt{b_1} + \sqrt{b_2}.$$

$$\sum_{k=0}^{3} \frac{1}{k+1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

$$\sum_{i=0}^{\infty} a_i = a_0 + a_1 + a_2 + a_3 + \ldots.$$

In some cases, you can evaluate the sum ("add up the terms"), so I got 55 for the value of the first sum. In other cases, you might leave the sum unevaluated (as in $\sum_{k=0}^{3} \frac{1}{k+1}$). What you do depends on the situation. In some cases, you can evaluate the sum using a computer or calculator. For example:

$$\sum_{n=1}^{50} \frac{1}{n} \approx 4.49920\ldots$$

It's often useful to go from a sum of terms written out "the long way" to a summation:

$$\sin(-2) + \sin(-1) + \sin 0 + \sin 1 + \sin 2 + \sin 3 = \sum_{n=-2}^{3} \sin n.$$

Notice that you can also write

$$\sin(-2) + \sin(-1) + \sin 0 + \sin 1 + \sin 2 + \sin 3 = \sum_{n=-1}^{4} \sin(n-1).$$

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There are infinitely many ways to write a sum using summation notation. It's a good habit to check that you have the correct range for your index by plugging in the top and bottom numbers. For example, using $\sum_{n=-1}^{4} \sin(n-1)$, I find that

$$n = -1 \quad \text{gives} \quad \sin(-1-1) = \sin(-2),$$

$$n = 4 \quad \text{gives} \quad \sin(4-1) = \sin 3.$$

These are the first and last terms for the original sum, so I have some assurance that I set it up correctly.

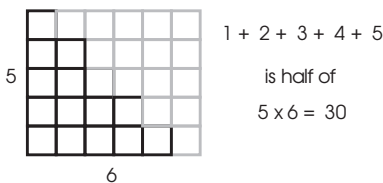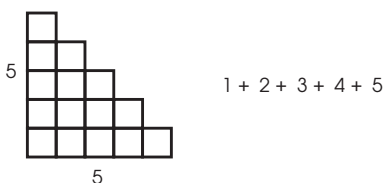Sometimes you can obtain a formula for a sum. Here are some important formulas.

Proposition.

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(2n+1)n(n+1)}{6}.$$

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}.$$

Proof. Here's a pictorial argument for the first formula:

Try drawing a picture for yourself in the case $n = 3$.

Here's the proof of the first formula by induction. If you're not familiar with induction, it's probably safe to skip this.

For $n = 1$, the left side is 1, and the right side is

$$\frac{1 \cdot (1+1)}{2} = 1.$$

This proves the result for $n = 1$.
Assume that the result holds for n:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

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I'll prove the result for $n + 1$:

$$1 + 2 + 3 + \cdots + n + (n + 1) = \frac{n(n+1)}{2} + (n+1)$$

$$= (n+1)\left(\frac{n}{2} + 1\right)$$

$$= (n+1)\frac{n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

This proves the result for $n + 1$, so it holds for all $n \geq 1$ by induction.
The other two formulas can also be proved by induction. I won't give the proofs here. □

Example. Evaluate the following sums:

(a) $\displaystyle\sum_{k=1}^{100} k$.

(b) $\displaystyle\sum_{k=1}^{100} k^2$.

(c) $\displaystyle\sum_{k=0}^{100} \left(\frac{1}{2}\right)^k$.

(a)
$$\sum_{k=1}^{100} k = 1 + 2 + 3 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050. \quad □$$

(b)
$$\sum_{k=1}^{100} k^2 = 1^2 + 2^2 + 3^2 + \cdots + 100^2 = \frac{201 \cdot 100 \cdot 101}{6} = 338350. \quad □$$

(c)
$$\sum_{k=0}^{100} \left(\frac{1}{2}\right)^k = \frac{1 - \left(\frac{1}{2}\right)^{101}}{1 - \frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^{100}. \quad □$$

Example. Find a closed-form expression for $\displaystyle\sum_{k=1}^{n}(3k^2 + 5k + 4)$.

$$\sum_{k=1}^{n}(3k^2 + 5k + 4) = 3\sum_{k=1}^{n} k^2 + 5\sum_{k=1}^{n} k + \sum_{k=1}^{n} 4 = 3 \cdot \frac{(2n+1)n(n+1)}{6} + 5 \cdot \frac{n(n+1)}{2} + 7n. \quad □$$

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Example. (Telescoping series) You can prove the following identity by combining the fractions on the right over a common denominator:

$$\frac{2}{k^2 + 2k} = \frac{1}{k} - \frac{1}{k+2}.$$

Use this identity to compute $\displaystyle\sum_{k=1}^{1000} \frac{2}{k^2 + 2k}$.

$$\sum_{k=1}^{1000} \frac{2}{k^2 + 2k} = \sum_{k=1}^{1000} \left(\frac{1}{k} - \frac{1}{k+2}\right) =$$

$$\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{999} - \frac{1}{1001}\right) + \left(\frac{1}{1000} - \frac{1}{1002}\right) =$$

$$\frac{1}{1} + \frac{1}{2} - \frac{1}{1001} - \frac{1}{1002} = \frac{751250}{501501}.$$

I used the given identity to rewrite the sum, then I wrote out enough of the terms that I could see what's happening. All the terms except for the four in the last line cancel: Each term from $\dfrac{1}{3}$ to $\dfrac{1}{1000}$ appears once with positive sign and once with negative sign. This leaves the four terms in the last line. ☐