

## Absolute Convergence and Conditional Convergence

The convergence tests I've discussed (such as the Ratio Test and Limit Comparison) apply to positive term series. What can you say about convergence if a series has negative terms?

If there are only finitely many negative terms, you can "chop them off" and consider the series that remains, which will have only positive terms. What about the case where there are infinitely many negative terms?

If the positive and negative terms alternate, the **Alternating Series Test** may tell you that the series converges. But there are series to which it does not apply.

One approach you might take to series with negative terms is to force all the negative terms to be positive by taking absolute values.

**Definition.** A series  $\sum_{k=1}^{\infty} a_k$  **converges absolutely** if the absolute value series  $\sum_{k=1}^{\infty} |a_k|$  converges.

Forcing all the terms to be positive should make it more difficult for a series to converge, since you lose the benefit of having negative terms cancelling with positive terms (which might keep the partial sums from blowing up). You would think that if you can do a more difficult thing (converge absolutely) then you ought to be able to do the easier thing (converge), and this turns out to be true.

**Theorem.** If  $\sum_{k=1}^{\infty} a_k$  converges absolutely, then it converges.

**Proof.** Suppose that  $\sum_{k=1}^{\infty} a_k$  converges absolutely, so  $\sum_{k=1}^{\infty} |a_k|$  converges.

**Step 1.**  $\sum_{k=1}^{\infty} (|a_k| - a_k)$  is a series with nonnegative terms.

If  $a_k \geq 0$ , then  $|a_k| = a_k$ , and  $|a_k| - a_k = 0$ . If  $a_k < 0$ , then  $|a_k| > a_k$  (because a positive number must be greater than a negative number), and so  $|a_k| - a_k > 0$ .

**Step 2.**  $\sum_{k=1}^{\infty} (|a_k| - a_k)$  converges.

By taking cases as in Step 1, I have  $|a_k| \geq -a_k$ . Adding  $|a_k|$  to both sides, I get  $2|a_k| \geq |a_k| - a_k$ . The series  $\sum_{k=1}^{\infty} |a_k|$  converges by assumption, so  $\sum_{k=1}^{\infty} 2|a_k|$  converges as well. Therefore, the inequality  $2|a_k| \geq |a_k| - a_k$  shows that  $\sum_{k=1}^{\infty} (|a_k| - a_k)$  converges by comparison.

**Step 3.**  $\sum_{k=1}^{\infty} a_k$  converges.

I have

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (|a_k| - (|a_k| - a_k)) = \sum_{k=1}^{\infty} |a_k| - \sum_{k=1}^{\infty} (|a_k| - a_k).$$

Since the two series on the right converge, it follows that  $\sum_{k=1}^{\infty} a_k$  converges as well.  $\square$

**Remark.** Here's another approach to the proof. The idea is to break the original series up into its positive and negative terms. Make new series as follows:

$$b_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n < 0 \end{cases} \quad \text{and} \quad c_n = \begin{cases} 0 & \text{if } a_n > 0 \\ a_n & \text{if } a_n < 0 \end{cases} .$$

Here's how they look in a particular case:

$$\begin{array}{rcccccccc} a_n : & 1 & -2 & -3 & 4 & 5 & -6 & \dots \\ b_n : & 1 & 0 & 0 & 4 & 5 & 0 & \dots \\ c_n : & 0 & -2 & -3 & 0 & 0 & -6 & \dots \end{array}$$

You can see that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n + c_n).$$

The series  $\sum_{n=1}^{\infty} b_n$  and  $\sum_{n=1}^{\infty} -c_n$  are series with nonnegative terms. By construction,

$$|a_n| \geq b_n \quad \text{and} \quad |a_n| \geq -c_n.$$

Since  $\sum_{n=1}^{\infty} |a_n|$  converges, the series  $\sum_{n=1}^{\infty} b_n$  and  $\sum_{n=1}^{\infty} -c_n$  both converge by direct comparison. Since  $\sum_{n=1}^{\infty} -c_n$  converges,  $\sum_{n=1}^{\infty} c_n$  converges as well.

Hence,  $\sum_{n=1}^{\infty} a_n$  converges, because it's the sum of  $\sum_{n=1}^{\infty} b_n$  and  $\sum_{n=1}^{\infty} c_n$ .

The theorem says something which is reasonable, and it's also useful: Sometimes the easiest way to show a series converges is to show that the absolute value series converges.

On the other hand, perhaps taking absolute values results in a series which no longer converges, even if the original series does.

**Definition.** A series  $\sum_{k=1}^{\infty} a_k$  **converges conditionally** if the absolute value series  $\sum_{k=1}^{\infty} |a_k|$  *diverges*, but the original series converges.

Note that to conclude that a series converges conditionally, you need to know two things:

(a) You need to know that the absolute value series **diverges** (so the original series doesn't converge absolutely).

(b) You need to know that the original series converges.

For example, consider the **alternating harmonic series**

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

I know it converges by the Alternating Series Test.

If I replace each term with its absolute value (removing the  $(-1)^{k+1}$ ), I get the **harmonic series**:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

The harmonic series diverges.

Therefore, the series  $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$  converges conditionally.

**Example.** Determine whether the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$  converges absolutely, converges conditionally, or diverges.

If I replace each term with its absolute value (removing the  $(-1)^{n+1}$ ), I get  $\sum_{n=1}^{\infty} \frac{1}{n^3}$ .

This is a  $p$ -series with  $p = 3 > 1$  so it converges.

Therefore, the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$  converges absolutely.  $\square$

Sometimes you can show a series converges by showing that it converges absolutely.

**Example.** Show that the following series converges:

$$\sum_{k=1}^{\infty} \frac{\sin k}{2^k} = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{2^3} + \cdots.$$

Note that the series does *not* alternate. In fact,

$$\sin 1 \approx 0.84147, \quad \sin 2 \approx 0.90930, \quad \sin 3 \approx 0.14112, \quad \sin 4 \approx -0.75680, \dots$$

Thus, you can't use the Alternating Series Test. On the other hand, since the series has negative terms, many convergence tests — the Integral Test, the Ratio Test, the Root Test — don't apply.

The trick is to consider the absolute value series, which is  $\sum_{k=1}^{\infty} \frac{|\sin k|}{2^k}$ . Since  $|\sin k| \leq 1$  for all  $k$ ,

$$\frac{|\sin k|}{2^k} \leq \frac{1}{2^k}.$$

The series  $\sum_{k=1}^{\infty} \frac{1}{2^k}$  is a convergent geometric series. Therefore, the series  $\sum_{k=1}^{\infty} \frac{|\sin k|}{2^k}$  converges by comparison.

Thus, the original series  $\sum_{k=1}^{\infty} \frac{\sin k}{2^k}$  converges absolutely. *If a series converges absolutely, then it converges.*

Hence, the series  $\sum_{k=1}^{\infty} \frac{\sin k}{2^k}$  converges.  $\square$

In problems which ask you to check for absolute or conditional convergence, you should be careful to do things in the correct order. Here's how to approach the question: "Does the series  $\sum_{k=1}^{\infty} a_k$  converge absolutely, converge conditionally, or diverge?"

1. Scan the series quickly and see if you can apply the Zero Limit Test. If  $\lim_{k \rightarrow \infty} a_k \neq 0$ , the series **diverges** (and that's all you have to do).

2. Check the absolute value series  $\sum_{k=1}^{\infty} |a_k|$  for convergence using your convergence tests for positive term series. If it converges, the original series **converges absolutely** and you can stop. If it diverges, go on to Step 3.

3. Now that you know the absolute value series diverges, you need to check for conditional convergence. Look at the original series  $\sum_{k=1}^{\infty} a_k$ . If it converges, you conclude that it **converges conditionally**; otherwise, it **diverges**.

**Remark.** There is no point in taking absolute values if the series has positive terms. For a series with positive terms, you simply check for convergence or divergence.

**Example.** Determine whether the series  $\sum_{k=1}^{\infty} (-1)^k \frac{2k^2 + 3}{3k^2 - 1}$  converges absolutely, converges conditionally, or diverges.

$$\lim_{k \rightarrow \infty} \frac{2k^2 + 3}{3k^2 - 1} = \frac{2}{3}, \quad \text{so} \quad \lim_{k \rightarrow \infty} (-1)^k \frac{2k^2 + 3}{3k^2 - 1} \text{ is undefined.}$$

The series diverges by the Zero Limit Test.  $\square$

**Example.** Determine whether the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^{1/2} + 1)(n^{1/3} + 1)}$  converges absolutely, converges conditionally, or diverges.

$$\lim_{n \rightarrow \infty} \frac{1}{(n^{1/2} + 1)(n^{1/3} + 1)} = 0.$$

The Zero Limit Test fails.

Consider the absolute value series  $\sum_{n=1}^{\infty} \frac{1}{(n^{1/2} + 1)(n^{1/3} + 1)}$ . (Taking absolute values removes the  $(-1)^n$ .)

For large  $n$ ,

$$\frac{1}{(n^{1/2} + 1)(n^{1/3} + 1)} \approx \frac{1}{n^{1/2} \cdot n^{1/3}} = \frac{1}{n^{5/6}}.$$

Apply Limit Comparison:

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{(n^{1/2} + 1)(n^{1/3} + 1)}}{\frac{1}{n^{5/6}}} = \lim_{n \rightarrow \infty} \frac{n^{5/6}}{(n^{1/2} + 1)(n^{1/3} + 1)} = 1.$$

The limit is finite and positive.

The series  $\sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$  diverges, because it's a  $p$ -series with  $p = \frac{5}{6} < 1$ . Therefore, the absolute value series diverges by Limit Comparison, and the original series does not converge absolutely.

Return to the original series. The terms alternate. If  $f(x) = \frac{1}{(x^{1/2} + 1)(x^{1/3} + 1)}$ , then

$$f'(x) = -\frac{\frac{1}{2}x^{-1/2}}{(x^{1/2} + 1)^2(x^{1/3} + 1)} - \frac{\frac{1}{3}x^{-2/3}}{(x^{1/2} + 1)(x^{1/3} + 1)^2}.$$

$f'(x) < 0$  for  $x \geq 1$ , so the terms decrease in magnitude. Finally,

$$\lim_{n \rightarrow \infty} \frac{1}{(n^{1/2} + 1)(n^{1/3} + 1)} = 0.$$

The hypotheses of the Alternating Series Test are satisfied, so the original series converges. Since the original series converges, but does not converge absolutely, it converges conditionally.  $\square$

**Example.** Determine whether the series  $\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^3 + 2}$  converges absolutely, converges conditionally, or diverges.

Consider the absolute value series  $\sum_{k=1}^{\infty} \frac{k}{k^3 + 2}$ . Apply Limit Comparison:

$$\lim_{k \rightarrow \infty} \frac{\frac{k}{k^3 + 2}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{k^3}{k^3 + 2} = 1.$$

The limit is a finite positive number. The series  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  converges, because it is a  $p$ -series with  $p = 2 > 1$ .

Therefore,  $\sum_{k=1}^{\infty} \frac{k}{k^3 + 2}$  converges by Limit Comparison.

Hence,  $\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^3 + 2}$  converges absolutely.  $\square$

**Example.** Determine whether the series  $\sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{3} + \cos \frac{\pi n}{4}}{n^2}$  converges absolutely, converges conditionally, or diverges.

The table below shows the sign of  $\sin \frac{\pi n}{3} + \cos \frac{\pi n}{4}$  for  $n = 1$  to  $n = 10$ .

$n$	1	2	3	4	5	6	7	8	9	10
sign of $\sin \frac{\pi n}{3} + \cos \frac{\pi n}{4}$	1	1	-1	-1	-1	0	1	1	1	-1

This is *not* an alternating series.

Consider the absolute value series  $\sum_{n=1}^{\infty} \frac{\left| \sin \frac{\pi n}{3} + \cos \frac{\pi n}{4} \right|}{n^2}$ .

$$\left| \sin \frac{\pi n}{3} + \cos \frac{\pi n}{4} \right| \leq 2, \quad \text{so} \quad \frac{\left| \sin \frac{\pi n}{3} + \cos \frac{\pi n}{4} \right|}{n^2} \leq \frac{2}{n^2}.$$

$\sum_{n=1}^{\infty} \frac{2}{n^2}$  converges, since it's a multiple of a  $p$ -series with  $p = 2 > 1$ .

Therefore, the absolute value series  $\sum_{n=1}^{\infty} \frac{\left| \sin \frac{\pi n}{3} + \cos \frac{\pi n}{4} \right|}{n^2}$  converges by comparison.  
Hence, the original series converges absolutely.  $\square$

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