4-14-2019

Arc Length

Suppose a curve is given by continuous functions

$$x = f(t), \quad y = g(t), \quad a \le\le b.$$

Partition the interval $[a, b]$:

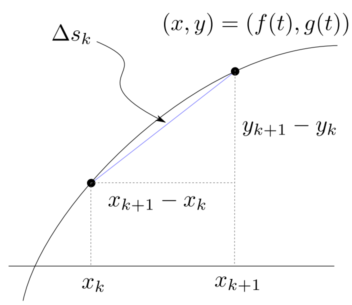
$$P: \ a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

(Note that different partitions may use different numbers of points, as well as different points.)
Consider a subinterval $[t_k, t_{k+1}]$. The corresponding points on the curve are

$$(x_k, y_k) = (f(t_k), g(t_k)) \quad \text{and} \quad (x_{k+1}, y_{k+1}) = (f(t_{k+1}), g(t_{k+1})).$$

The length of the segment from (x_k, y_k) to (x_{k+1}, y_{k+1}) is

$$\Delta s_k = \sqrt{(x_k - x_{k+1})^2 + (y_k - y_{k+1})^2}.$$



It approximates the length of the curve from (x_k, y_k) to (x_{k+1}, y_{k+1}).
For the partition P, the total length of the segments is

$$L(P) = \sum_{k=0}^{n-1} \Delta s_k.$$

Definition. A curve is **rectifiable** if there is a number M such that for every partition of the interval $[a, b]$,

$$L(P) < M.$$

If a curve is rectifiable, we can define the **length** of the curve as the least upper bound of $L(P)$ taken over all the partitions of the interval.

While you can imagine *approximating* the length of a curve by taking partitions with larger and larger numbers of points, this definition doesn't give a way of computing the exact length.

If the curve is "well-behaved", we can compute the exact length as follows. Suppose the functions $f(t)$ and $g(t)$ are differentiable and have continuous derivatives. Apply the Mean Value Theorem to f and to g on a typical subinterval $[t_k, t_{k+1}]$. Then there are numbers p_k and q_k such that

$$x_{k+1} - x_k = f'(p_k)(t_{k+1} - t_k) \quad \text{and} \quad y_{k+1} - y_k = g'(q_k)(t_{k+1} - t_k).$$

Plugging these into the equation for Δs_k above, I get

$$\begin{aligned}\Delta s_k &= \sqrt{(x_k - x_{k+1})^2 + (y_k - y_{k+1})^2} \\ &= \sqrt{f'(p_k)^2(t_{k+1} - t_k)^2 + g'(q_k)^2(t_{k+1} - t_k)^2} \\ &= \sqrt{f'(p_k)^2 + g'(q_k)^2}(t_{k+1} - t_k) \\ &= \sqrt{f'(p_k)^2 + g'(q_k)^2}\Delta t_k\end{aligned}$$

I obtain the sum

$$L(P) = \sum_{k=0}^{n-1} \Delta s_k = \sum_{k=0}^{n-1} \sqrt{f'(p_k)^2 + g'(q_k)^2}\Delta t_k.$$

I want to take the limit as the number of subintervals in the partition becomes infinite (or as the length of the subintervals goes to 0). There is a technical point here, and that is that I have *two* varying quantities p_k and q_k, so this is not an ordinary Riemann sum. In fact, it's possible to show (using a result called Bliss's Theorem) that the Riemann sum produces the expected definite integral:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt = \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \sqrt{f'(p_k)^2 + g'(q_k)^2}\Delta t_k.$$

This gives the length of the curve. You can also write this in the form

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

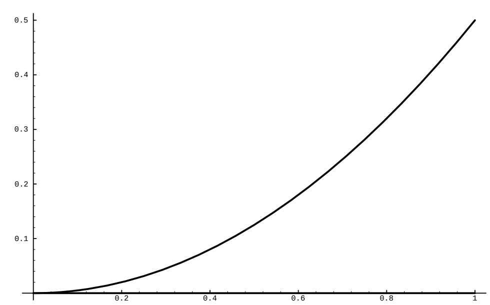
If the curve is given in the form $y = g(x)$, we can think of it as parametrized by x (so t becomes x). Since $\dfrac{dx}{dx} = 1$, the formula is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Likewise, if the curve is given in the form $x = f(y)$, the formula is

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$

Example. Find the length of $y = \dfrac{1}{2}x^2$ for $0 \le x \le 1$.



$$\frac{dy}{dx} = x, \quad \text{so} \quad \left(\frac{dy}{dx}\right)^2 + 1 = x^2 + 1.$$

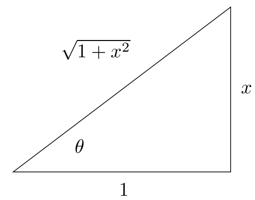
The length is

$$L = \int_0^1 \sqrt{x^2+1}\,dx = \left[\frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|x + \sqrt{x^2+1}|\right]_0^1 = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2}) = 1.14779\ldots.$$

Here's the work for the integral:

$$\int \sqrt{x^2+1}\,dx = \int \sqrt{(\tan\theta)^2 + 1}(\sec\theta)^2\,d\theta = \int (\sec\theta)^3\,d\theta =$$

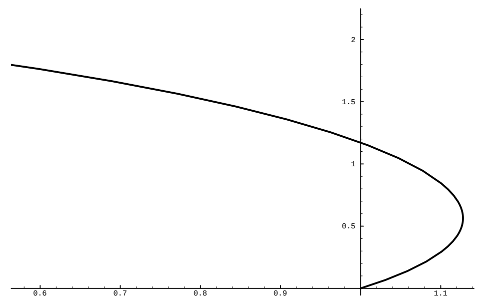
$$\left[x = \tan\theta, \quad dx = (\sec\theta)^2\,d\theta\right]$$



$$\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|x + \sqrt{x^2+1}| + C. \quad \square$$

Example. Find the length of the curve

$$x = e^t \cos 2t, \quad y = e^t \sin 2t, \quad 0 \le t \le \frac{\pi}{4}.$$



$$\frac{dx}{dt} = -2e^t \sin 2t + e^t \cos 2t, \quad \frac{dy}{dt} = 2e^t \cos 2t + e^t \sin 2t,$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2e^t \sin 2t + e^t \cos 2t)^2 + (2e^t \cos 2t + e^t \sin 2t)^2 =$$

$$4e^{2t}(\sin 2t)^2 - 4e^{2t}\sin 2t\cos 2t + e^{2t}(\cos 2t)^2 + 4e^{2t}(\cos 2t)^2 + 4e^{2t}\sin 2t\cos 2t + e^{2t}(\sin 2t)^2 =$$

$$4e^{2t}\left[(\sin 2t)^2 + (\cos 2t)^2\right] + e^{2t}\left[(\sin 2t)^2 + (\cos 2t)^2\right] = 4e^{2t}\cdot 1 + e^{2t}\cdot 1 = 5e^{2t}.$$

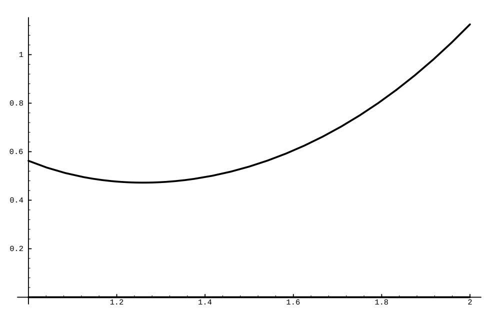
Hence,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{5e^{2t}} = \sqrt{5}e^t.$$

The length is

$$L = \int_0^{\pi/4} \sqrt{5}\,e^t\,dt = \sqrt{5}\left[e^t\right]_0^{\pi/4} = \sqrt{5}(e^{\pi/4} - 1) = 2.66825\ldots. \quad \square$$

Example. Find the length of $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2}$ for $1 \le x \le 2$.



$$\frac{dy}{dx} = \frac{x^3}{4} - \frac{1}{x^3}, \quad \text{so} \quad \left(\frac{dy}{dx}\right)^2 = \left(\frac{x^3}{4} - \frac{1}{x^3}\right)^2 = \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6}.$$

The next step is the algebraic trick in this problem:

$$\left(\frac{dy}{dx}\right)^2 + 1 = \left(\frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6}\right) + 1 = \frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} = \left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2.$$

The idea is that I saw when I found $\left(\frac{dy}{dx}\right)^2$ that

$$\frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6} = \left(\frac{x^3}{4} - \frac{1}{x^3}\right)^2.$$

Therefore,

$$\frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} = \left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2.$$

The only difference is in the sign of the $\frac{1}{2}$. Since the first expression is the square of a binomial with a "$-$", the second expression must be the square of the same binomial with a "$+$".

Thus,

$$\sqrt{\left(\frac{dy}{dx}\right)^2 + 1} = \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2} = \frac{x^3}{4} + \frac{1}{x^3}.$$

The length is

$$L = \int_1^2 \left(\frac{x^3}{4} + \frac{1}{x^3}\right) dx = \left[\frac{x^4}{16} - \frac{1}{2x^2}\right]_1^2 = \frac{7}{16} = 0.4375. \quad \square$$
