Direct and Limit Comparison

You can often tell that a series converges or diverges by comparing it to a known series. I'll look first at situations where you can establish an *inequality* between the terms of two series.

Theorem. (**Direct Comparison**) Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, be series with positive terms.

(a) If $a_k \ge b_k$ for all k and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} b_k$ converges.

(b) If $a_k \ge b_k$ for all k and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

Proof. Let's look at the proof of (a). I know that $\sum_{k=1}^{\infty} a_k$ converges; say $\sum_{k=1}^{\infty} a_k = S$.

The partial sums of $\sum_{k=1}^{\infty} a_k$ increase, since the series has positive terms. Therefore, the partial sums are bounded above by S.

Since $a_k \ge b_k$ for all k, the partial sums of $\sum_{k=1}^{\infty} a_k$ are greater than or equal to the partial sums of $\sum_{k=1}^{\infty} b_k$:

$$a_1 + a_2 + \cdots + a_n \ge b_1 + b_2 + \cdots + b_n.$$

Hence, S is an upper bound for the partial sums of $\sum_{k=1}^{\infty} b_k$. Since those partial sums form an increasing sequence that is bounded above, they must have a limit. This means that $\sum_{k=1}^{\infty} b_k$ converges.

A similar idea works for (b). In that case, the a_k partial sums are always bigger than the b_k partial sums, but the b_k partial sums go to ∞. Hence, the a_k partial sums go to ∞ as well. □

In the problems that follow, I'll often have to establish inequalities involving fractions. I need to know how a fraction changes if its top or its bottom is made bigger or smaller. The following table summarizes the ideas:

	top bigger	top smaller	bottom bigger	bottom smaller
fraction becomes ...	bigger	smaller	smaller	bigger

For example, take the fraction $\dfrac{2}{3}$. If I change the top from "2" to "3", I make the top bigger. The fraction changes from $\dfrac{2}{3}$ to $\dfrac{3}{3}$, so the fraction has become bigger. If I change the bottom from "3" to "2", I make the bottom smaller. The fraction changes from $\dfrac{2}{3}$ to $\dfrac{2}{2}$, so the fraction has become bigger.

Example. Determine whether $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3 + k + 7}$ converges or diverges.

The series has positive terms. In fact, I could use the Integral Test, but who would want to integrate $\dfrac{1}{x^3 + x + 7}$?

Instead, note that when k is large, the k^3 term should dominate. How does $\dfrac{1}{k^3+k+7}$ compare to $\dfrac{1}{k^3}$? Well, if you make the bottom *smaller*, the fraction gets *bigger*:

$$\frac{1}{k^3+k+7} < \frac{1}{k^3}.$$

Now $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a p-series with $p = 3$, so it converges. Hence, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+k+7}$ converges by comparison. □

Example. Determine whether $\displaystyle\sum_{n=3}^{\infty} \frac{6n^2+1}{5n^3-2}$ converges or diverges.

When n is large, the term $\dfrac{6n^2+1}{5n^3-2}$ is approximately $\dfrac{6n^2}{5n^3} = \dfrac{6}{5}\dfrac{1}{n}$. This is a term of the harmonic series, which diverges. So I suspect my series diverges.

I have

$$\frac{6n^2+1}{5n^3-2} > \frac{6n^2}{5n^3-2} > \frac{6n^2}{5n^3} = \frac{6}{5}\frac{1}{n}.$$

I made the top smaller, then I made the bottom bigger; both cause the fraction to become smaller.

$\displaystyle\sum_{n=3}^{\infty} \frac{6}{5}\frac{1}{n}$ is $\dfrac{6}{5}$ times the harmonic series (minus the $n = 1$ and $n = 2$ terms), so it diverges. Hence, $\displaystyle\sum_{n=3}^{\infty} \frac{6n^2+1}{5n^3-2}$ diverges by Direct Comparison. □

Example. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{5\sin n + 11}{6^n+4}$ converges or diverges.

Since the top is bounded and the bottom is approximately 6^n for large n, the series terms "look like" $\dfrac{1}{6^n}$, which is the general term of a convergent geometric series. So I think my series converges.

I'll use a familiar fact from trigonometry, then do algebra to "build up" the term of my series in the inequality.

$$\begin{aligned} -1 \le&\ \sin n \le 1 \\ -5 \le&\ 5\sin n \le 5 \\ 6 \le&\ 5\sin n + 11 \le 16 \\ \frac{6}{6^n+4} \le&\ \frac{5\sin n + 11}{6^n+4} \le \frac{16}{6^n+4} < \frac{16}{6^n} \end{aligned}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{16}{6^n}$ is geometric with ratio $\dfrac{1}{6}$, so it converges. Hence, the original series converges by direct comparison. □

Example. Determine whether $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{k^3}$ converges or diverges.

The series has positive terms. Since $\arctan k \le \dfrac{\pi}{2}$,

$$\frac{\arctan k}{k^3} \le \frac{\pi}{2}\frac{1}{k^3}.$$

$\displaystyle\sum_{k=1}^{\infty} \frac{\pi}{2}\frac{1}{k^3}$ converges, because it's a p-series with $p = 3 > 1$. Therefore, $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{k^3}$ converges by direct comparison. $\square$

Example. Determine whether $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+3}}{k\sqrt{k+2}}$ converges or diverges.

If you make the top smaller, the fraction gets smaller:

$$\frac{\sqrt{k+3}}{k\sqrt{k+2}} > \frac{\sqrt{k+2}}{k\sqrt{k+2}} = \frac{1}{k}.$$

Notice how I avoided changing $k+3$ to k; I changed it to something which cancelled the radical on the bottom.

Now $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges since it's harmonic. So $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+3}}{k\sqrt{k+2}}$ diverges, by comparison. $\square$

Note that Direct Comparison won't work if the inequalities go the wrong way. For example, consider $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2-2}$. I'd like to compare this to $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$, but if I make the bottom *bigger* (by adding 2), the fraction gets *smaller*:

$$\frac{1}{k^2-2} > \frac{1}{k^2}.$$

It's true that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series ($p = 2$), but it's *smaller* than the given series. I can't draw a conclusion this way.

Nevertheless, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ *is* "close to" the given series in some sense. **Limit Comparison** will make precise the idea that one series is "close to" another, without having to worry about inequalities.

Theorem. (**Limit Comparison**) Let $\displaystyle\sum_{k=1}^{\infty} a_k$ be a positive term series. Let $\displaystyle\sum_{k=1}^{\infty} b_k$ be a positive term series whose behavior is known.

Consider the limiting ratio

$$\lim_{k\to\infty} \frac{a_k}{b_k}.$$

(a) If the limit is a *finite positive number*, then the two series behave in the same way:

(i) If $\displaystyle\sum_{k=1}^{\infty} b_k$ converges, then $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\displaystyle\sum_{k=1}^{\infty} b_k$ diverges, then $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges.

(b) If the limit is 0 and $\displaystyle\sum_{k=1}^{\infty} b_k$ converges, then $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

(c) If the limit is $+\infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

The first case is the most important one, and the one I'll usually use. Fortunately, it will work even if you accidentally write $\dfrac{b_k}{a_k}$ instead of $\dfrac{a_k}{b_k}$. The second and third cases require that you get the fraction "right side up". The phrase "finite positive number" means that in case (a), the limit should not be 0 or ∞.

Proof. I'll sketch a proof in the first part of case (a). Suppose the limit is a finite positive number:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0.$$

Suppose $\sum_{k=1}^{\infty} b_k$ converges. Then there is a number n such that if $k \ge n$,

$$\frac{a_k}{b_k} < L + 1.$$

So for $k \ge n$,

$$a_k < (L+1)b_k.$$

Apply Direct Comparison to the series $\sum_{k=n}^{\infty} a_k$ and $\sum_{k=n}^{\infty} (L+1)b_k$. The series $\sum_{k=n}^{\infty} (L+1)b_k$ converges, because $\sum_{k=1}^{\infty} b_k$ converges. Hence, $\sum_{k=n}^{\infty} a_k$ converges by Direct Comparison.

Therefore, $\sum_{k=1}^{\infty} a_k$ converges as well. □

Example. Determine whether $\sum_{k=1}^{\infty} \dfrac{4k^2 + k + 9}{7k^3 + 13}$ converges or diverges.

The series has positive terms.
When k is large, the top and bottom are dominated by the terms with the biggest powers:

$$\frac{4k^2 + k + 9}{7k^3 + 13} \approx \frac{4k^2}{7k^3} = \frac{4}{7} \cdot \frac{1}{k}.$$

Compute the limiting ratio:

$$\lim_{k\to\infty} \frac{\dfrac{4k^2 + k + 9}{7k^3 + 13}}{\dfrac{4}{7} \cdot \dfrac{1}{k}} = \frac{7}{4} \cdot \lim_{k\to\infty} \frac{4k^3 + k^2 + 9k}{7k^3 + 13} = \frac{7}{4} \cdot \frac{4}{7} = 1.$$

The limiting ratio is 1, a finite positive number. The series $\sum_{k=1}^{\infty} \dfrac{4}{7} \cdot \dfrac{1}{k}$ diverges, because it is harmonic. Hence, the series $\sum_{k=1}^{\infty} \dfrac{4k^2 + k + 9}{7k^3 + 13}$ diverges by Limit Comparison. □

Example. Determine whether $\sum_{k=2}^{\infty} \dfrac{4^k + 5}{7^k - 42}$ converges or diverges.

The series has positive terms.
When k is large,

$$\frac{4^k + 5}{7^k - 42} \approx \frac{4^k}{7^k}.$$

Compute the limiting ratio:

$$\lim_{k\to\infty} \frac{\dfrac{4^k + 5}{7^k - 42}}{\dfrac{4^k}{7^k}} = \lim_{k\to\infty} \frac{1 + \dfrac{5}{4^k}}{1 - \dfrac{42}{7^k}} = 1.$$

The limiting ratio is 1, a finite positive number. The series $\displaystyle\sum_{k=1}^{\infty} \frac{4^k}{7^k}$ is a convergent geometric series (since $\dfrac{4}{7} < 1$). Therefore, $\displaystyle\sum_{k=2}^{\infty} \frac{4^k + 5}{7^k - 42}$ converges, by Limit Comparison. □
