Infinite Series - Review

Example. Does the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+1.01^n\right)$ converge or diverge?

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p=2>1$, so it converges. $\sum_{n=1}^{\infty}1.01^n$ is a geometric series with $r=1.01>1$, so it diverges.

Hence, the sum of the two series *diverges*. □

Example. Determine whether the series converges or diverges. If it converges, find its sum.

$$\frac{7}{16}-\frac{7}{64}+\frac{7}{256}-\cdots+(-1)^n(7)\left(\frac{1}{4}\right)^n+\cdots$$

The series is geometric with ratio $-\frac{1}{4}$, so it converges. The sum is

$$\frac{7}{16}-\frac{7}{64}+\frac{7}{256}-\cdots+(-1)^n\left(\frac{1}{7}\right)\left(\frac{1}{4}\right)^n+\cdots=\frac{7}{16}\left(1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\cdots\right)=$$

$$\frac{7}{16}\cdot\frac{1}{1-\left(-\frac{1}{4}\right)}=\frac{7}{16}\cdot\frac{4}{5}=\frac{7}{20}.\quad □$$

Example. Does the series $\sum_{n=1}^{\infty}\frac{\sqrt{e^n}}{\sqrt[3]{\pi^n}}$ converge or diverge?

$$\sum_{n=1}^{\infty}\frac{\sqrt{e^n}}{\sqrt[3]{\pi^n}}=\sum_{n=1}^{\infty}\left(\frac{\sqrt{e}}{\sqrt[3]{\pi}}\right)^n.$$

The series is geometric with ratio $\frac{\sqrt{e}}{\sqrt[3]{\pi}}\approx 1.12572>1$. Therefore, the series diverges. □

Example. Does the series $\sum_{n=1}^{\infty}\frac{4n^3+5}{7n^2-11n^3}$ converge or diverge?

$$\lim_{n\to\infty}\frac{4n^3+5}{7n^2-11n^3}=-\frac{4}{11}\neq 0.$$

The series diverges by the Zero Limit Test. □

Example. Does the series $\sum_{n=1}^{\infty}\tan^{-1}\frac{42^n}{41^n}$ converge or diverge?

$\frac{42^n}{41^n}$ is a geometric *sequence* with ratio $\frac{42}{41}>1$, so $\frac{42^n}{41^n}\to\infty$ as $n\to\infty$. Therefore,

$$\lim_{n\to\infty}\tan^{-1}\frac{42^n}{41^n}=\frac{\pi}{2}\neq 0.$$

Hence, the series diverges, by the Zero Limit Test. □

Example. Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converge or diverge?

The terms are positive. The function $f(x) = \dfrac{1}{x(\ln x)^2}$ is continuous for $x \geq 2$. The derivative is

$$f'(x) = -\frac{1}{x^2(\ln x)^2} - \frac{2}{x^2(\ln x)^3}.$$

I have $f'(x) < 0$ for $x \geq 2$. Thus, the terms of the series decrease. The hypotheses of the Integral Test are satisfied.

Compute the integral:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{x \cdot u^2} \cdot x\,du = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} =$$

$$\left[u = \ln x, \quad du = \frac{dx}{x}, \quad dx = x\,du; \quad x = 2, \quad u = \ln 2; \quad x = b, \quad u = \ln b\right]$$

$$\lim_{b\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln b} = \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

The integral converges, so the series converges, by the Integral Test. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1 + e^{-n^3}}{n}$ converge or diverge?

Notice that as $n \to \infty$, $1 + e^{-n^3}$ *decreases* to 1. Thus,

$$1 + e^{-n^3} \geq 1, \quad \text{so} \quad \frac{1 + e^{-n^3}}{n} \geq \frac{1}{n}.$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic, so it diverges. Therefore, the original series diverges by direct comparison. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}}$ converge or diverge?

For large n, $\dfrac{\sqrt{n(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}} \approx \dfrac{\sqrt{n^2}}{\sqrt{n^3}} = \dfrac{1}{n^{1/2}}$. Do a Limit Comparison:

$$\lim_{n\to\infty} \frac{\dfrac{\sqrt{n(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}}}{\dfrac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{\sqrt{n(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}} \frac{n^{1/2}}{1} = \lim_{n\to\infty} \frac{\sqrt{n^2(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}} =$$

$$\sqrt{\lim_{n\to\infty} \frac{n^2(n+1)}{(n+2)(n+3)(n+4)}} = \sqrt{1} = 1.$$

The limit is a finite positive number. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, because it's a p-series with $p = \dfrac{1}{2} < 1$. Therefore, the original series diverges, by Limit Comparison. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{26^n (n!)^3}{(3n)!}$ converge or diverge?

Apply the Ratio Test:

$$\lim_{n\to\infty} \frac{\dfrac{26^{n+1}((n+1)!)^3}{(3n+3)!}}{\dfrac{26^n (n!)^3}{(3n)!}} = \lim_{n\to\infty} \frac{26^{n+1}((n+1)!)^3}{(3n+3)!} \cdot \frac{(3n)!}{26^n (n!)^3} = \lim_{n\to\infty} \frac{26^{n+1}}{26^n} \cdot \frac{((n+1)!)^3}{(n!)^3} \cdot \frac{(3n)!}{(3n+3)!} =$$

$$\lim_{n\to\infty} 26 \cdot \left(\frac{(n+1)!}{n!}\right)^3 \cdot \frac{(3n)!}{(3n+3)!} =$$

$$\lim_{n\to\infty} 26 \cdot \left(\frac{(1)(2)\cdots(n)(n+1)}{(1)(2)\cdots(n)}\right)^3 \cdot \frac{(1)(2)\cdots(3n)}{(1)(2)\cdots(3n)(3n+1)(3n+2)(3n+3)} =$$

$$\lim_{n\to\infty} 26 \cdot (n+1)^3 \cdot \frac{1}{(3n+1)(3n+2)(3n+3)} = \lim_{n\to\infty} \frac{26(n+1)^3}{(3n+1)(3n+2)(3n+3)} = \frac{26}{27} < 1.$$

Therefore, the series converges by the Ratio Test. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \left(\sin^{-1} \frac{1}{n}\right)^n$ converge or diverge?

Apply the Root Test:

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(\sin^{-1} \frac{1}{n}\right) = \sin^{-1} 0 = 0 < 1.$$

The series converges by the Root Test. □

Example. The series $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{\sqrt[3]{k+5}}$ converges by the Alternating Series Test.

Find the smallest value of n for which the partial sum $\displaystyle\sum_{k=1}^{n} (-1)^{k+1} \frac{2}{\sqrt[3]{k+5}}$ approximates the actual value of the sum to within 0.01.

The error in using $\displaystyle\sum_{k=1}^{n} (-1)^{k+1} \frac{2}{\sqrt[3]{k+5}}$ to approximate the actual value of the sum is less than the

$(n+1)^{\text{st}}$ term in absolute value, so I want

$$\begin{aligned}
\frac{2}{\sqrt[3]{(n+1)+5}} &< 0.01 \\
\frac{2}{\sqrt[3]{n+6}} &< 0.01 \\
2 &< 0.01\sqrt[3]{n+6} \\
200 &< \sqrt[3]{n+6} \\
8\,000\,000 &< n+6 \\
7\,999\,994 &< n
\end{aligned}$$

The smallest value of n is $n = 7\,999\,995$. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5n+2}$ converge absolutely, converge conditionally, or diverge?

Consider the absolute value series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n+2}$. By Limit Comparison,

$$\lim_{n\to\infty} \frac{\dfrac{1}{5n+2}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+2} = \frac{1}{5}.$$

The limit is a finite positive number. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, because it's harmonic. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n+2}$ diverges by Limit Comparison.

Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5n+2}$ does not converge absolutely.

Consider the original series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5n+2}$. It alternates, and if $f(n) = \dfrac{1}{5n+2}$,

$$f'(n) = -\frac{5}{(5n+2)^2} < 0 \quad \text{for} \quad n \geq 1.$$

Hence, the terms decrease in absolute value. Finally,

$$\lim_{n\to\infty} \frac{1}{5n+2} = 0.$$

By the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5n+2}$ converges. Since it converges, but does not converge absolutely, it converges conditionally. □
