1-17-2019

Integration Review

In this section, I'll review some of our integration techniques.

Example. Compute $\displaystyle\int \ln(1+x^2)\,dx$.

Use integration by parts:

$$\begin{array}{ccccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & \ln(1+x^2) & & 1 \\ & & \searrow & \\ - & \dfrac{2x}{x^2+1} & \rightarrow & x \end{array}$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - 2\int \frac{x^2}{x^2+1}\,dx = x\ln(1+x^2) - 2\int \left(1 - \frac{1}{x^2+1}\right)dx =$$

$$x\ln(1+x^2) - 2x + 2\tan^{-1} x + C. \quad \square$$

Example. Compute $\displaystyle\int \frac{7x^3-2x^2+12-8}{x^2(x^2+4)}\,dx$.

Use partial fractions:

$$\frac{7x^3-2x^2+12-8}{x^2(x^2+4)} = \frac{ax+b}{x^2+4} + \frac{c}{x} + \frac{d}{x^2},$$

$$7x^3-2x^2+12x-8 = (ax+b)x^2 + cx(x^2+4) + d(x^2+4).$$

Let $x=0$. Then $-8=4d$, so $d=-2$. Therefore,

$$7x^3-2x^2+12x-8 = (ax+b)x^2 + cx(x^2+4) - 2(x^2+4).$$

Differentiate:

$$21x^2-4x+12 = (ax+b)(2x) + ax^2 + cx(2x) + c(x^2+4) - 4x.$$

Let $x=0$. Then $12=4c$, so $c=3$. Therefore,

$$21x^2-4x+12 = (ax+b)(2x) + ax^2 + 3x(2x) + 3(x^2+4) - 4x,$$

$$21x^2-4x+12 = (ax+b)(2x) + ax^2 + 6x^2 + 3(x^2+4) - 4x.$$

Differentiate:

$$42x-4 = (ax+b)(2) + a(2x) + a(2x) + 12x + 6x - 4.$$

Let $x=0$. Then $-4=2b-4$, so $b=0$. Therefore,

$$42x-4 = (ax)(2) + a(2x) + a(2x) + 12x + 6x - 4.$$

Let $x=1$. Then $38=6a+14$, so $a=4$.
Therefore,

$$\int \frac{7x^3-2x^2+12-8}{x^2(x^2+4)}\,dx = \int \left(\frac{4x}{x^2+4} + \frac{3}{x} - \frac{2}{x^2}\right)dx = 2\ln|x^2+4| + 3\ln|x| + \frac{2}{x} + C. \quad \square$$

Example. Compute $\int (\cos 3x)^4 \, dx$.

I'll use the double angle formula

$$(\cos\theta)^2 = \frac{1}{2}(1 + \cos 2\theta).$$

Begin by writing the fourth power as the square of a square, then use the formula:

$$\int (\cos 3x)^4 \, dx = \int \left((\cos 3x)^2\right)^2 \, dx = \int \left(\frac{1}{2}(1 + \cos 6x)\right)^2 \, dx = \frac{1}{4}\int \left(1 + 2\cos 6x + (\cos 6x)^2\right) \, dx =$$

$$\frac{1}{4}\int \left(1 + 2\cos 6x + \frac{1}{2}(1 + \cos 12x)\right) \, dx = \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 6x + \frac{1}{2}\cos 12x\right) \, dx =$$

$$\frac{3}{8}x + \frac{1}{12}\sin 6x + \frac{1}{96}\sin 12x + C. \quad \square$$

Example. Compute $\int (\cos 3x)^4 (\sin 3x)^3 \, dx$.

Since one of the powers of sine or cosine is odd, I do *not* use a double angle formula.

$$\int (\cos 3x)^4 (\sin 3x)^3 \, dx = \int (\cos 3x)^4 (\sin 3x)^2 (\sin 3x \, dx) = \int (\cos 3x)^4 \left(1 - (\cos 3x)^2\right) (\sin 3x \, dx) =$$

$$\left[u = \cos 3x, \quad du = -3\sin 3x \, dx, \quad dx = \frac{du}{-3\sin 3x}\right]$$

$$\int u^4 (1 - u^2)(\sin 3x) \left(\frac{du}{-3\sin 3x}\right) = \frac{1}{3}\int (u^6 - u^4) \, du = \frac{1}{3}\left(\frac{1}{7}u^7 - \frac{1}{5}u^5\right) + C =$$

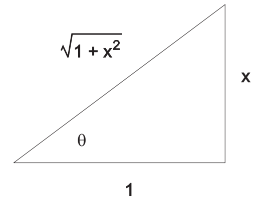
$$\frac{1}{3}\left(\frac{1}{7}(\cos 3x)^7 - \frac{1}{5}(\cos 3x)^5\right) + C. \quad \square$$

Example. Compute $\int \frac{\sqrt{1 + x^2}}{x^4} \, dx$.

Use trig substitution:

$$\int \frac{\sqrt{1 + x^2}}{x^4} \, dx = \int \frac{\sqrt{1 + (\tan\theta)^2}}{(\tan\theta)^4} (\sec\theta)^2 \, d\theta = \int \frac{\sec\theta}{(\tan\theta)^4} (\sec\theta)^2 \, d\theta = \int \frac{(\sec\theta)^3}{(\tan\theta)^4} \, d\theta =$$

$$\left[x = \tan\theta, \quad dx = (\sec\theta)^2 \, d\theta\right]$$



$$\int \frac{1}{(\cos\theta)^3} \cdot \frac{(\cos\theta)^4}{(\sin\theta)^4} \, d\theta = \int \frac{\cos\theta}{(\sin\theta)^4} \, d\theta = \int \frac{\cos\theta}{u^4} \cdot \frac{du}{\cos\theta} = \int \frac{du}{u^4} =$$

$$\left[u = \sin\theta, \quad du = \cos\theta \, d\theta, \quad d\theta = \frac{du}{\cos\theta}\right]$$

$$-\frac{1}{3u^3}+C=-\frac{1}{3(\sin\theta)^3}+C=-\frac{1}{3}\cdot\frac{(1+x^2)^{3/2}}{x^3}+C.\quad\square$$

Example. Compute $\displaystyle\int\frac{1}{x+x^{7/9}}\,dx$.

Let $x=u^9$, so $dx=9u^8\,du$.

$$\int\frac{1}{x+x^{7/9}}\,dx=\int\frac{9u^8}{u^9+u^7}\,du=9\int\frac{u}{u^2+1}\,du=9\int\frac{u}{w}\cdot\frac{dw}{2u}=\frac{9}{2}\int\frac{1}{w}\,dw=$$

$$\left[w=u^2+1,\quad dw=2u\,du,\quad du=\frac{dw}{2u}\right]$$

$$\frac{9}{2}\ln|w|+C=\frac{9}{2}\ln|u^2+1|+C=\frac{9}{2}\ln|x^{2/9}+1|+C.$$

For the last step, I used the fact that $x=u^9$ gives $u=x^{1/9}$, so $u^2=x^{2/9}$. $\square$
