3-31-2019

Intervals of Convergence of Power Series

A **power series** is an infinite series

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

The number c is called the **expansion point**.

A power series may *represent* a function $f(x)$, in the sense that wherever the series converges, it converges to $f(x)$. There are two issues here:

1. Where does the series converge?

2. If the series converges at a point, does it converge to $f(x)$?

For example, consider the series

$$\sum_{n=0}^{\infty} u^n = 1 + u + u^2 + u^3 + \cdots + u^n + \cdots.$$

Since the terms of the series involve powers of the variable u (i.e. $u - 0$), the expansion point is $c = 0$.

This series represents the function $\dfrac{1}{1-u}$ for $-1 < u < 1$. You can see that this is reasonable by dividing 1 by $1-u$, or by using the the formula for the sum of a geometric series with ratio $r = u$.

For example, if $u = \dfrac{1}{2}$,

$$\frac{1}{1-u} = 2, \quad \text{while} \quad 1 + u + u^2 + u^3 + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \cdots.$$

Results on geometric series show that the two expressions are equal.

On the other hand, if $u = -1$,

$$\frac{1}{1-u} = \frac{1}{2}, \quad \text{while} \quad 1 + u + u^2 + u^3 + \cdots = 1 - 1 + 1 - 1 + 1 - \cdots.$$

The two expressions *aren't* equal; in fact, the series on the right diverges, by the Zero Limit Test.

You can use the Ratio Test (and sometimes, the Root Test) to determine the values for which a power series converges. Here are some important facts about the convergence of a power series.

(a) A power series converges absolutely in a symmetric interval about its expansion point, and diverges outside that symmetric interval. The distance from the expansion point to an endpoint is called the **radius of convergence**.

(b) Any combination of convergence or divergence may occur at the endpoints of the interval. That is, the series may diverge at both endpoints, converge at both endpoints, or diverge at one and converge at the other.

(c) A power series always converges at the expansion point.

The set of points where the series converges is called the **interval of convergence**.

For example, here is a power series expanded around $c = 5$:

$$a_0 + a_1(x-5) + a_2(x-5)^2 + a_3(x-5)^2 + \cdots.$$

It surely converges at $x = 5$, since setting $x = 5$ gives $a_0 + 0 + 0 + \cdots = a_0$.

The series converges on an interval which is *symmetric* about $a = 5$. Thus, $-2 < x < 12$ is a possible interval of convergence; $3 < x < 8$ is not.

Suppose you know that $3 < x < 7$ is the largest open interval on which the series converges. Then the series can do anything (in terms of convergence or divergence) at $x = 3$ and $x = 7$. The interval of convergence could be $3 < x < 7$ (diverges at both ends), $3 \le x \le 7$ (converges at both ends), or $3 \le x < 7$ or $3 < x \le 7$ (converges at one end and diverges at the other).

Example. Determine the interval of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n}$.

Take absolute values and apply the Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\dfrac{|x|^{n+1}}{3^{n+1}}}{\dfrac{|x|^n}{3^n}} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \frac{|x|}{3}.$$

The series converges for $\dfrac{|x|}{3} < 1$. Solving the inequality, I get $|x| < 3$, or $-3 < x < 3$. The series diverges for $x < -3$ and for $x > 3$.

I'll test the endpoints separately.

At $x = 3$, the series is

$$\sum_{n=0}^{\infty} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots.$$

The series diverges at $x = 3$.

At $x = -3$, the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - \cdots.$$

The series diverges at $x = -3$.

All together, the series diverges for $x \le -3$ and for $x \ge 3$. It converges for $-3 < x < 3$.

You would reach the same conclusion using the Root Test:

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{|x|^n}{3^n}\right)^{1/n} = \frac{|x|}{3}.$$

The Root Test says that the series converges for $\dfrac{|x|}{3} < 1$, i.e. for $-3 < x < 3$, and that it diverges for $x < -3$ and for $x > 3$. The endpoint check is the same as above. $\square$

Example. Determine the interval of convergence for the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 5^n}$.

Take absolute values and apply the Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\dfrac{|x-2|^{n+1}}{(n+1)\cdot 5^{n+1}}}{\dfrac{|x-2|^n}{n \cdot 5^n}} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{|x-2|^{n+1}}{|x-2|^n} = \lim_{n\to\infty} \frac{1}{5}\frac{n}{n+1}|x-2| = \frac{1}{5}|x-2|.$$

By the Ratio Test, the series converges (absolutely) for $\dfrac{1}{5}|x-2| < 1$, or $-3 < x < 7$. Likewise, the series diverges for $x < -3$ or for $x > 7$.

Check the situation at the endpoints. For $x = 7$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, and it diverges.

For $x = -3$, the series is $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$. This is the alternating harmonic series, and it converges by the Alternating Series Test.

To summarize, the series converges absolutely for $-3 < x < 7$, converges conditionally for $x = -3$, and diverges for $x < -3$ and for $x \geq 7$. The interval of convergence is $-3 \leq x < 7$. □

Example. Determine the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Take absolute values and apply the Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

The limit is less than 1, *independent* of the value of x. It follows that the series converges for all x. That is, the interval of convergence is $-\infty < x < +\infty$.

In fact, this series represents the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \quad \square$$

Example. Determine the interval of convergence for the series $\sum_{n=0}^{\infty} n^n x^n$.

Take absolute values and apply the Root Test:

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (n^n |x|^n)^{1/n} \lim_{n\to\infty} n|x| \to \infty.$$

This means that the series diverges for all x except $x = 0$, the point of expansion. At $x = 0$, the series looks like

$$\sum_{n=0}^{\infty} n^n 0^n = \sum_{n=0}^{\infty} 0 = 0.$$

So it certainly converges for $x = 0$. □

Example. Determine the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{2^{2n} x^n}{n^2}$.

Take absolute values and apply the Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{2^{2n+2}|x|^{n+1}}{(n+1)^2}}{\frac{2^{2n}|x|^n}{n^2}} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n^2}{(n+1)^2} \cdot \frac{2^{2n+2}}{2^{2n}} = \lim_{n\to\infty} 4|x| \cdot \frac{n^2}{(n+1)^2} = 4|x|.$$

The series converges for $4|x| < 1$, i.e. for $-\dfrac{1}{4} < x < \dfrac{1}{4}$. The series diverges for $x < -\dfrac{1}{4}$ and for $x > \dfrac{1}{4}$. I'll test the endpoints separately.

At $x = \dfrac{1}{4}$, the series is

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{4^n \cdot n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The series is a p-series with $p = 2 > 1$, so it converges.

At $x = -\dfrac{1}{4}$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{4^n \cdot n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

This series converges absolutely, since the absolute value series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p-series with $p = 2 > 1$. Hence, it converges.

All together, the power series converges for $-\dfrac{1}{4} \le x \le \dfrac{1}{4}$, and diverges for $x < -\dfrac{1}{4}$ and for $x > \dfrac{1}{4}$. □

Example. Determine the interval of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{n+1}$.

Take absolute values and apply the Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\dfrac{|x+3|^{n+1}}{n+2}}{\dfrac{|x+3|^n}{n+1}} = \lim_{n\to\infty} \frac{|x+3|^{n+1}}{|x+3|^n} \cdot \frac{n+1}{n+2} = \lim_{n\to\infty} |x+3| \cdot \frac{n+1}{n+2} = |x+3|.$$

The series converges for $|x+3| < 1$, i.e. for $-4 < x < -2$, and diverges for $x < -4$ and for $x > -2$. I'll test the endpoints separately.

At $x = -2$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$. This is the harmonic series, so it diverges.

At $x = -4$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$. This is the Alternating Harmonic Series, so it converges.

All together, the power series converges for $-4 \le x < 2$, and diverges for $x < -4$ and for $x \ge -2$. □

Example. Determine the interval of convergence for the series

$$\frac{\ln 3}{3} x^3 + \frac{\ln 4}{4} x^4 + \cdots + \frac{\ln n}{n} x^n + \cdots.$$

Take absolute values and apply the Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\dfrac{\ln(n+1)}{n+1}|x|^{n+1}}{\dfrac{\ln n}{n}|x|^n} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \frac{n+1}{n} \cdot |x|.$$

By L'Hôpital's Rule,

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{\dfrac{1}{n+1}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \frac{n+1}{n} \cdot |x| = |x|.$$

The series converges for $|x| < 1$, i.e. for $-1 < x < 1$, and diverges for $x < -1$ and for $x > 1$.
I'll test the endpoints separately.

At $x = 1$, the series is $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$. The terms are positive; if $f(n) = \dfrac{\ln n}{n}$, then

$$f'(n) = -\frac{\ln n}{n^2} + \frac{1}{n^2} = \frac{1 - \ln n}{n^2} < 0 \quad \text{for} \quad n \ge 3.$$

Thus, the terms decrease. Apply the Integral Test:

$$\int_3^{\infty} \frac{\ln x}{x}\, dx = \lim_{b\to\infty} \int_3^b \frac{\ln x}{x}\, dx = \lim_{b\to\infty} \left[\frac{1}{2}(\ln x)^2\right]_3^b = \lim_{b\to\infty} \frac{1}{2}\left((\ln b)^2 - (\ln 3)^2\right) = +\infty.$$

Here's the work for the integral:

$$\int \frac{\ln x}{x}\, dx = \int \frac{u}{x} \cdot x\, du = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

$$\left[u = \ln x, \quad du = \frac{dx}{x}, \quad dx = x\, du\right]$$

Since the integral diverges, the series diverges, by the Integral Test.

At $x = -1$, the series is $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{\ln n}{n}$. The terms alternate, and the computation above shows that the terms decrease in absolute value. Finally, by L'Hôpital's Rule,

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0.$$

By the Alternating Series Test, the series converges.
All together, the series converges for $-1 \le x < 1$, and diverges for $x < -1$ and for $x \ge 1$. □

Example. Find the function represented by the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}(x-1)^n$ at the points where the series converges.

The series is geometric with ratio $-\dfrac{1}{5}(x-1)$, so

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}(x-1)^n = \frac{1}{1 - \left(-\frac{1}{5}(x-1)\right)} = \frac{1}{1 + \frac{1}{5}(x-1)} = \frac{5}{5 + (x-1)} = \frac{5}{4+x}. \quad □$$
