Miscellaneous Integration Techniques

In this section, I'll look at some other integration techniques.

Completing the square.

When an integral contains a quadratic expression ax^2+bx+c — that is, a quadratic with a middle term bx — you can sometimes simplify the integrand by **completing the square**. This eliminates the middle term of the quadratic; the resulting integral can then be computed using (e.g.) trig substitution.

Example. Compute $\displaystyle\int \frac{1}{x^2+4x+13}\,dx$.

Since $\dfrac{1}{2}\cdot 4 = 2$ and $2^2 = 4$, I have

$$x^2+4x+13 = x^2+4x+4+9 = (x+2)^2+9.$$

Therefore,

$$\int \frac{1}{x^2+4x+13}\,dx = \int \frac{1}{(x+2)^2+9}\,dx = \int \frac{1}{u^2+9}\,du =$$

$$\left[u = x+2, \quad du = dx\right]$$

$$\frac{1}{3}\tan^{-1}\frac{u}{3} + C = \frac{1}{3}\tan^{-1}\frac{x+2}{3} + C. \quad \square$$

Example. Compute $\displaystyle\int \frac{\sqrt{x^2+2x-8}}{x+1}\,dx$.

Since $\dfrac{1}{2}\cdot 2 = 1$ and $1^2 = 1$, I have

$$x^2+2x-8 = x^2+2x+1-9 = (x+1)^2-9.$$

Therefore,

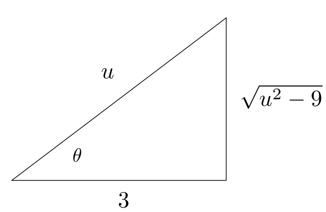
$$\int \frac{\sqrt{x^2+2x-8}}{x+1}\,dx = \int \frac{\sqrt{(x+1)^2-9}}{x+1}\,dx = \int \frac{\sqrt{u^2-9}}{u}\,du.$$

$$\left[u = x+1, \quad du = dx\right]$$

Next, I need a trig substitution:

$$\int \frac{\sqrt{u^2-9}}{u}\,du = \int \frac{\sqrt{9(\sec\theta)^2-9}}{3\sec\theta}(3\sec\theta\tan\theta\,d\theta) = \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta\,d\theta) = 3\int (\tan\theta)^2\,d\theta =$$

$$\left[x = 3\sec\theta, \quad dx = 3\sec\theta\tan\theta\,d\theta\right]$$



$$3\int \left((\sec\theta)^2 - 1\right)\,d\theta = 3\left(\tan\theta - \theta\right) + C = 3\left(\frac{\sqrt{u^2-9}}{3} - \sec^{-1}\frac{u}{3}\right) + C =$$

$$3\left(\frac{\sqrt{(x+1)^2-9}}{3} - \sec^{-1}\frac{x+1}{3}\right) + C. \quad \square$$

Note that in some cases, an integral containing a quadratic with a middle term can be integrated in other ways. For example, in this integral I can let $u = x - 3$:

$$\int \frac{1}{x^2-6x+9}\,dx = \int \frac{1}{(x-3)^2}\,dx = -\frac{1}{x-3} + C.$$

This integral can be done using partial fractions:

$$\int \frac{1}{x^2-5x+6}\,dx = \int \frac{1}{(x-2)(x-3)}\,dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right)\,dx = \ln|x-3| - \ln|x-2| + C.$$

Fractional powers.

When an integral contains fractional powers $x^{m/n}$, you can often simplify the integrand using a substitution of the form

$$x = u^k.$$

Take k to be the least common multiple of the denominators of the fractions that occur in the exponent.

Example. Compute $\displaystyle\int \frac{dx}{x^{3/4}+x^{2/3}}$.

Since the least common multiple of the denominators 3 and 4 is 12, I'll use $x = u^{12}$:

$$\int \frac{dx}{x^{3/4}+x^{2/3}} = \int \frac{12u^{11}\,du}{u^9+u^8} = 12\int \frac{u^3}{u+1}\,du.$$

$$\left[x = u^{12}, \quad dx = 12u^{11}\,du\right]$$

Since the top has a higher power than the bottom, I do a long division:

$$\begin{array}{r|l} & u^2 - u + 1 \\ \hline u + 1 & u^3 \\ & -\ (u^3 + u^2) \\ \hline & -u^2 \\ & -\ (-u^2 - u) \\ \hline & u \\ & -\ (u + 1) \\ \hline & -1 \end{array}$$

This gives

$$\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}.$$

Hence, my integral is

$$12\int\left(u^2 - u + 1 - \frac{1}{u+1}\right)\,du =$$

$$12\left(\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1|\right) + C = 12\left(\frac{1}{3}x^{1/4} - \frac{1}{2}x^{1/6} + x^{1/12} - \ln|x^{1/12}+1|\right) + C. \quad \square$$

Example. Compute $\displaystyle\int \frac{dx}{x^{1/2}(x^{1/2}+3x^{1/5})}$.

Since the least common multiple of the denominators 2 and 5 is 10, I'll use $x = u^{10}$:

$$\int \frac{dx}{x^{1/2}(x^{1/2}+3x^{1/5})} = \int \frac{10u^9\,du}{u^5(u^5+3u^2)} = 10\int \frac{u^2}{u^3+3}\,du = 10\int \frac{u^2}{w}\cdot\frac{dw}{3u^2} = \frac{10}{3}\int \frac{dw}{w} = \frac{10}{3}\ln|w| + C =$$

$$\left[x = u^{10}, \quad dx = 10u^9\,du\right] \qquad \left[w = u^3+3, \quad dw = 3u^2\,du, \quad du = \frac{dw}{3u^2}\right]$$

$$\frac{10}{3}\ln|u^3+3| + C = \frac{10}{3}\ln|x^{3/10}+3| + C. \quad \square$$

Example. Compute $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$.

The idea here is the same as in the last two examples: I can eliminate a square root by putting a square inside.

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u^2-1}{\sqrt{u^2}}\cdot 2u\,du = \int \frac{u^2-1}{u}\cdot 2u\,du = 2\int (u^2-1)\,du = 2\left(\frac{1}{3}u^3 - u\right) + C =$$

$$\left[x+1 = u^2, \quad dx = 2u\,du; \quad x = u^2-1, \quad u = (x+1)^{1/2}\right]$$

$$\frac{2}{3}(x+1)^{3/2} - 2\sqrt{x+1} + C. \quad \square$$

Example. Compute $\displaystyle\int \cos\sqrt{x}\,dx$.

Let $x = u^2$, so $dx = 2u\,du$. I get

$$\int \cos\sqrt{x}\,dx = 2\int u\cos u\,du.$$

I'll do this integral by parts:

	$\frac{d}{du}$	$\int du$
$+$	u	$\cos u$
$-$	1	$\sin u$
$+$	0	$-\cos u$

I do the parts computation and put the x's back using $u = \sqrt{x}$ (since $x = u^2$):

$$\int \cos\sqrt{x}\,dx = 2\int u\cos u\,du = 2\left(u\sin u + \cos u\right) + C = 2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + C. \quad \square$$
