4-18-2020

Parametric Equations of Curves

The **parametric equations** for a curve in the plane consists of a pair of equations

$$x = f(t), \quad y = g(t), \quad a \le t \le b.$$

Each value of the **parameter** t gives values for x and y; the point (x, y) is the corresponding point on the curve.

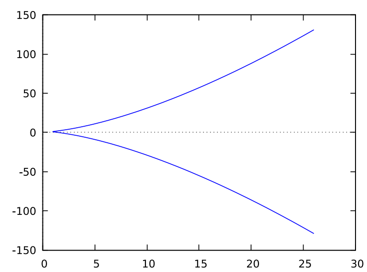
For example, consider the parametric equations

$$x = t^2 + 1, \quad y = t^3 + t + 1.$$

Here are some points (x, y) which result from plugging in some values for t:

t	x	y
-2	5	-9
-1	2	-1
0	1	1
1	2	3
2	5	11

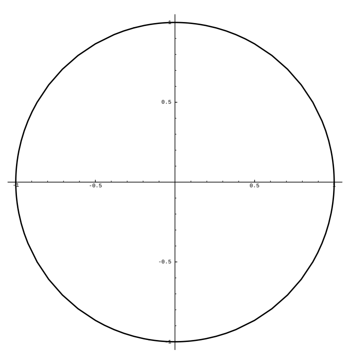
The graph of the curve looks like this:



$x = t^2 + 1$, $y = t^3 + t + 1$

These are parametric equations for the circle $x^2 + y^2 = 1$:

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$$



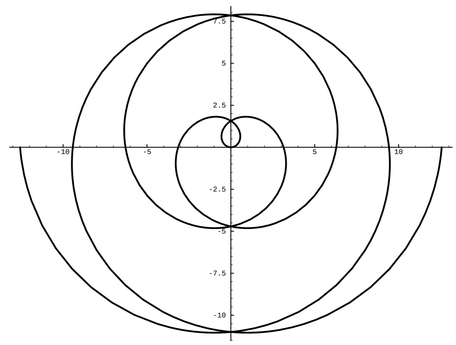
You can sometimes recover the x-y equation of a parametric curve by eliminating t from the parametric equations. In this case,

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1.$$

Notice that the graph of a circle is *not* the graph of a function. Parametric equations can represent more general curves than function graphs can, which is one of their advantages.

These parametric equations represent a spiral:

$$x = t\cos t, \quad y = t\sin t,$$



This is also not the graph of a function $y = f(x)$.

Example. Find the x-y equation for

$$x = t^3 + 1, \quad y = t^2 + t + 1.$$

Solve the x-equation for t:

$$\begin{aligned} x &= t^3 + 1 \\ x - 1 &= t^3 \\ (x-1)^{1/3} &= t \end{aligned}$$

Plug this expression for t into the y-equation:

$$y = (x-1)^{2/3} + (x-1)^{1/3} + 1. \quad \square$$

Example. Find the x-y equation for

$$x = 5 + 2\cos t, \quad y = -3 + \sin t.$$

Notice that

$$\cos t = \frac{x-5}{2}, \quad \sin t = y + 3.$$

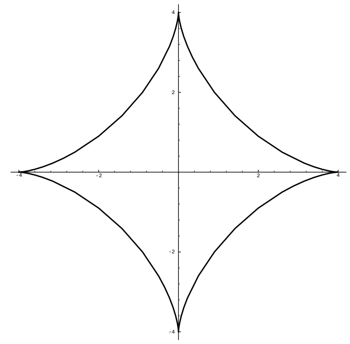
So

$$\left(\frac{x-5}{2}\right)^2 + (y+3)^2 = (\cos t)^2 + (\sin t)^2 = 1.$$

This is the standard form for the equation of an ellipse. $\square$

In some cases, recovering an x-y equation would be difficult or impossible. For example, these are the parametric equations for a **hypocycloid of four cusps**:

$$x = 3\cos t + \cos 3t, \quad y = 3\sin t - \sin 3t, \quad 0 \le t \le 2\pi$$



In this case, it would be difficult to eliminate t to obtain an x-y equation.

What about going the other way? If you have a curve (or an x-y equation), how do you obtain parametric equations?

Note first that a given curve can be represent by infinitely many sets of parametric equations. For example, all of these sets of parametric equations represent the unit circle $x^2 + y^2 = 1$:

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi.$$

$$x = \cos 11t, \quad y = \sin 11t, \quad 0 \le t \le \frac{2\pi}{11}.$$

$$x = -\sin t, \quad y = \cos t, \quad 0 \le t \le 2\pi.$$

Even so, it can be difficult to find parametrizations for curves.

Let's start with an easy case. If you have x-y equations in which x or y is solved for, it's easy. For example, to parametrize $y = x^2$, set $x = t$. Then $y = x^2 = t^2$. A parametrization is given by

$$x = t, \quad y = t^2.$$

To parametrize $x = 3y - y^2$, set $y = t$. Then $x = 3y - y^2 = 3t - t^2$, so

$$x = 3t - t^2, \quad y = t.$$

This is a parametrization of $x = 3y - y^2$. (This is how you can graph x-in-terms-of-y equations on your calculator.)

Here's another important case. If (a, b) and (c, d) are points, the **line** through (a, b) and (c, d) may be parametrized by

$$x = a + t(c - a), \quad y = b + t(d - b), \quad -\infty < t < \infty.$$

It is easiest to remember this in the **vector form**

$$(x, y) = (1 - t) \cdot (a, b) + t \cdot (c, d).$$

Notice that when $t = 0$, I have $(x, y) = (a, b)$, and when $t = 1$, $(x, y) = (c, d)$. Thus, if you let $0 \le t \le 1$, you get the **segment** from (a, b) to (c, d).

Example. Find parametric equations for the line through $(3, -6)$ and $(5, 2)$.

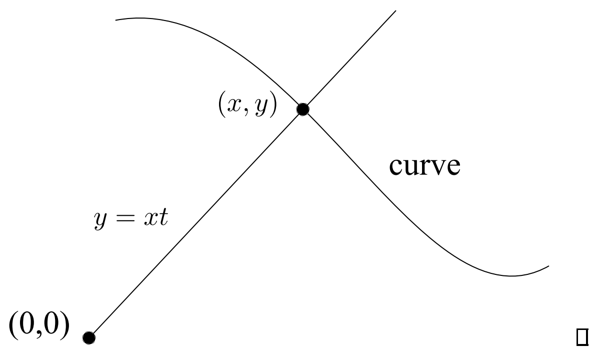
$$\begin{aligned}(x, y) &= (1 - t) \cdot (3, -6) + t \cdot (5, 2) \\ &= (3 - 3t, -6 + 6t) + (5t, 2t) \\ &= (3 + 2t, -6 + 8t)\end{aligned}$$

Thus,

$$x = 3 + 2t \quad \text{and} \quad y = -6 + 8t. \quad \square$$

An analogous result holds for lines in 3 dimensions (or in any number of dimensions).

As an example of a more general method of parametrizing curves, I'll consider **parametrizing by slope**. The idea is to think of a point (x, y) on the curve as the intersection point of the curve and the line $y = xt$:



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The slope t will be the parameter for the curve.

Example. Find parametric equations for the **Folium of Descartes**:

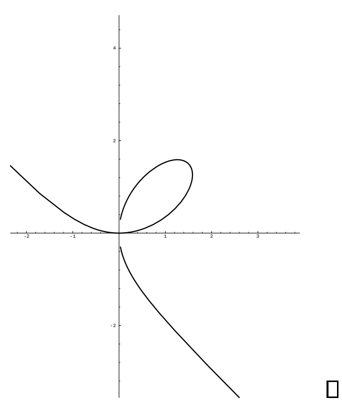
$$x^3 + y^3 = 3xy.$$

Set $y = xt$. Then

$$\begin{aligned} x^3 + x^3t^3 &= 3x^2t \\ x + xt^3 &= 3t \\ x(1 + t^3) &= 3t \\ x &= \frac{3t}{1 + t^3} \end{aligned}$$

Therefore, $y = xt = \dfrac{3t^2}{1 + t^3}$. A parametrization is given by

$$x = \frac{3t}{1 + t^3}, \quad y = \frac{3t^2}{1 + t^3}.$$



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The first and second derivatives give information about the shape of a curve. Here's how to find the derivatives for a parametric curve.

First, by the Chain Rule,

$$\frac{dx}{dt} \cdot \frac{dy}{dx} = \frac{dy}{dt}.$$

Solving for $\dfrac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$$

Example. Find the equation of the tangent line to the curve

$$x = e^t + t^2, \quad y = e^{2t} + 3t \quad \text{at} \quad t = 1.$$

$$\frac{dy}{dx} = \frac{2e^{2t} + 3}{e^t + 2t}, \quad \text{so} \quad \left.\frac{dy}{dx}\right|_{t=1} = \frac{2e^2 + 3}{e + 2}.$$

When $t = 1$, $x = e + 1$ and $y = e^2 + 3$. The tangent line is

$$y - (e^2 + 3) = \left(\frac{2e^2 + 3}{e + 2}\right)(x - (e + 1)). \quad \square$$

Example. At what points on the following curve is the tangent line horizontal?

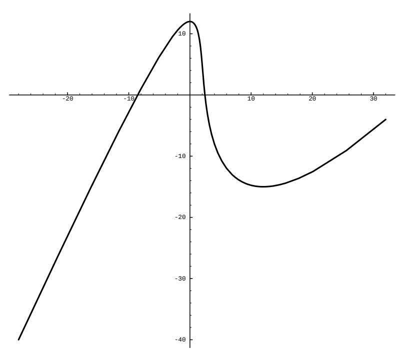
$$x = t^3 + t + 2, \quad y = 2t^3 - 3t^2 - 12t + 5$$

Find the derivative:

$$\frac{dy}{dx} = \frac{6t^2 - 6t - 12}{3t^2 + 1} = \frac{6(t - 2)(t + 1)}{3t^2 + 1}.$$

The tangent line is horizontal when $\dfrac{dy}{dx} = 0$, and $\dfrac{dy}{dx} = 0$ for $t = 2$ and for $t = -1$.

When $t = 2$, $x = 12$ and $y = -15$. When $t = -1$, $x = 0$ and $y = 12$. There are horizontal tangents are $(12, -15)$ and at $(0, 12)$.



$\square$

To find the second derivative, I differentiate the first derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Since $\frac{dy}{dx}$ will come out in terms of t, I want to be sure to differentiate $\frac{dy}{dx}$ with respect to t. Use the Chain Rule again:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx} \cdot \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right] = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

That is,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Example. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = 1$ for

$$x = t^2 + t + 2, \quad y = 2t^3 - t + 5.$$

First,

$$\frac{dx}{dt} = 2t + 1, \quad \frac{dy}{dt} = 6t^2 - 1.$$

So

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2 - 1}{2t + 1}.$$

When $t = 1$, I have $\frac{dy}{dx} = \frac{5}{3}$.

Next,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\frac{6t^2 - 1}{2t + 1} = \frac{(2t + 1)(12t) - (6t^2 - 1)(2)}{(2t + 1)^2}.$$

So

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{(2t + 1)(12t) - (6t^2 - 1)(2)}{(2t + 1)^2}}{2t + 1} = \frac{(2t + 1)(12t) - (6t^2 - 1)(2)}{(2t + 1)^3}.$$

When $t = 1$, I have $\frac{d^2y}{dx^2} = \frac{26}{27}$. □
