Power Series - Review

Example. Expand $f(x) = \dfrac{5}{11-x}$ in a power series at $c = 3$ and find the interval of convergence.

I will use the series for $\dfrac{1}{1-u}$:

$$\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots = \sum_{n=0}^{\infty} u^n.$$

I need powers of $x - 3$, so I make an "$x - 3$" on the bottom, then fix the numbers so the value of the fraction doesn't change. Then I do algebra to put my function into the form $\dfrac{1}{1-u}$, at which point I can substitute:

$$\frac{5}{11-x} = \frac{5}{8-(x-3)} = 5 \cdot \frac{1}{8-(x-3)} = \frac{5}{8} \cdot \frac{1}{1 - \frac{1}{8}(x-3)} =$$

$$\frac{5}{8}\left(1 + \frac{1}{8}(x-3) + \frac{1}{8^2}(x-3)^2 + \cdot\right) = \frac{5}{8} \sum_{n=0}^{\infty} \frac{1}{8^n}(x-3)^n.$$

I substituted $u = \dfrac{1}{8}(x-3)$ in the u-series to get my series.

The interval of convergence for the series for $\dfrac{1}{1-u}$ is $-1 < u < 1$. Substitute $u = \dfrac{1}{8}(x-3)$:

$$\begin{aligned} -1 &< \frac{1}{8}(x-3) < 1 \\ -8 &< x - 3 < 8 \\ -5 &< x < 11 \end{aligned} \qquad \square$$

Example. Find the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{3^n (x-5)^n}{2^n}$.

Apply the Root Test:

$$\lim_{n \to \infty} \left(\frac{3^n |x-5|^n}{2^n}\right)^{1/n} = \lim_{n \to \infty} \frac{3}{2}|x-5| = \frac{3}{2}|x-5|.$$

The series converges for

$$\frac{3}{2}|x-5| < 1, \quad \text{i.e. for} \quad \frac{13}{3} < x < \frac{17}{3}.$$

At $x = \dfrac{17}{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{3^n}{2^n}\left(\frac{17}{3} - 5\right)^n = \sum_{n=0}^{\infty} 1.$$

This series diverges by the Zero Limit Test.

At $x = \dfrac{13}{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{3^n}{2^n}\left(\frac{13}{3} - 5\right)^n = \sum_{n=0}^{\infty} (-1)^n.$$

This series also diverges by the Zero Limit Test.

The power series converges for $\frac{13}{3} < x < \frac{17}{3}$ and diverges elsewhere. □

Example. Expand $f(x) = e^{-3x}$ in a power series at $c = 4$ and find the interval of convergence.

$$e^{-3x} = e^{-3(x-4)-12} = e^{-12}e^{-3(x-4)}.$$

Set $u = -3(x-4)$ in

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots.$$

This gives

$$e^{-3x} = e^{-12}\left(1 - 3(x-4) + \frac{3^2(x-4)^2}{2!} - \frac{3^3(x-4)^3}{3!} + \cdots\right).$$

The interval of convergence for the e^u series is $-\infty < u < \infty$. So for the e^{-3x} series,

$$-\infty < -3(x-4) < +\infty, \quad -\infty < x < +\infty. \quad \square$$

Example. Expand $(\cos 5x)^2$ in a Taylor series at $c = 0$.

Using the double angle formula

$$(\cos 5x)^2 = \frac{1}{2}(1 + \cos 10x) = \frac{1}{2} + \frac{1}{2}\cos 10x = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{10^2x^2}{2!} + \frac{10^4x^4}{4!} - \frac{10^6x^6}{6!} + \cdots\right) =$$

$$1 - \frac{1}{2}\frac{10^2x^2}{2!} + \frac{1}{2}\frac{10^4x^4}{4!} - \frac{1}{2}\frac{10^6x^6}{6!} + \cdots. \quad \square$$

Example. (a) Use the first four nonzero terms of the Taylor series for e^u at $c = 0$ to approximate $\int_0^1 e^{-x^3}\,dx$.

(b) Use the Alternating Series Test to estimate the error in part (a).

(a)

$$e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} - \cdots.$$

Hence,

$$\int_0^1 e^{-x^3}\,dx = \int_0^1 \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} - \cdots\right) dx = \left[x - \frac{x^4}{4} + \frac{x^7}{14} - \frac{x^{10}}{60} + \frac{x^{13}}{312} - \cdots\right]_0^1 =$$

$$1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312} - \cdots \approx 0.80476.$$

(I used the first four terms to get the approximation.) □

(b) The error is no greater than the next term, which is $\frac{1}{312} = 0.00320\ldots$. □

Example. Use the Taylor series expansion of $\sin x$ at $c = 0$ to explain the fact that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

The series for $\sin x$ at $c = 0$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Divide by x to obtain

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots.$$

Then

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) = 1. \quad \Box$$

Example. Find the first four nonzero terms of the Taylor expansion for $y = \sin x$ at $c = 3$.

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x.$$

The series is

$$\sin x = \sin 3 + (\cos 3)(x - 3) - \frac{\sin 3}{2!}(x - 3)^2 - \frac{\cos 3}{3!}(x - 3)^3 + \cdots. \quad \Box$$

Example. Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x - 5)^{2n}}{n \cdot 9^n}$.

$$\lim_{n \to \infty} \frac{\dfrac{|x - 5|^{2n+2}}{(n + 1) \cdot 9^{n+1}}}{\dfrac{|x - 5|^{2n}}{n \cdot 9^n}} = \lim_{n \to \infty} \frac{|x - 5|^{2n+2}}{(n + 1) \cdot 9^{n+1}} \frac{n \cdot 9^n}{|x - 5|^{2n}} =$$

$$\lim_{n \to \infty} \frac{9^n}{9^{n+1}} \cdot \frac{n}{n + 1} \cdot \frac{|x - 5|^{2n+2}}{|x - 5|^{2n}} = \lim_{n \to \infty} \frac{n}{n + 1} \cdot \frac{1}{9}|x - 5| = \frac{1}{9}|x - 5|^2.$$

The series converges for

$$\frac{1}{9}|x - 5|^2 < 1, \quad \text{i.e. for} \quad 2 < x < 8.$$

At $x = 8$, the series is

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{n \cdot 9^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is harmonic, so it diverges.
At $x = 2$, the series is

$$\sum_{n=1}^{\infty} \frac{(-3)^{2n}}{n \cdot 9^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is harmonic, so it diverges.
The power series converges for $2 < x < 8$ and diverges elsewhere. $\Box$

Example. $f(x)$ satisfies

$$f(2) = 5, \quad f'(2) = 2, \quad f''(2) = -3, \quad f'''(2) = 12.$$

Use the third degree Taylor polynomial for f at $c = 2$ to approximate $f(2.1)$.

The third degree Taylor polynomial for f at $c = 2$ is

$$p_3(x) = 5 + 2(x-2) - \frac{3}{2!}(x-2)^2 + \frac{12}{3!}(x-2)^3 = 5 + 2(x-2) - \frac{3}{2}(x-2)^2 + 2(x-2)^3.$$

So

$$f(2.1) \approx p_3(2.1) = 5 + (2)(0.1) - \left(\frac{3}{2}\right)(0.1)^2 + 2 \cdot (0.1)^3 = 5.187. \quad \square$$

Example. Suppose that $f^{(5)}(x) = \dfrac{2}{(1-x)^7}$. Use $R_4(x;0)$ to estimate the error in using the fourth degree Taylor polynomial at $c = 0$ to approximate $f(x)$ for $0 \le x \le 0.1$.

For some z between 0 and x,

$$R_4(x;0) = \frac{f^{(5)}(z)}{5!}x^5 = \frac{1}{120}\left(\frac{2}{(1-z)^7}\right)x^5.$$

Since $0 \le x \le 0.1$, $x^5 \le 0.1^5$.
For the z-term, I have $0 \le z \le x \le 0.1$. Thus,

$$\begin{aligned}
0 \le& z \le 0.1 \\
0 \ge& -z \ge -0.1 \\
1 \ge& 1 - z \ge 0.9 \\
1 \ge& (1-z)^7 \ge 0.9^7 \\
1 \le& \frac{1}{(1-z)^7} \le \frac{1}{0.9^7} \\
2 \le& \frac{2}{(1-z)^7} \le \frac{2}{0.9^7}
\end{aligned}$$

So $\dfrac{2}{(1-z)^7} \le \dfrac{2}{0.9^7}$. Therefore,

$$R_4(x;0) \le \left(\frac{1}{120}\right)\left(\frac{2}{0.9^7}\right)(0.1^5) \approx 3.4845 \cdot 10^{-7}. \quad \square$$
